

SHORT MATURITY ASIAN OPTIONS FOR THE CEV MODEL

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ABSTRACT. We present a rigorous study of the short maturity asymptotics for Asian options with continuous-time averaging, under the assumption that the underlying asset follows the Constant Elasticity of Variance (CEV) model. We present an analytical approximation for the Asian options prices which has the appropriate short maturity asymptotics, and demonstrate good numerical agreement of the asymptotic results with the results of Monte Carlo simulations and benchmark test cases for option parameters relevant in practical applications.

1. INTRODUCTION

Asymptotics for option prices and implied volatility of European options for the short maturity regime have been extensively studied in the literature, see e.g. [8, 36, 37, 40, 14] for local volatility models, [28, 50, 5, 6] for the exponential Lévy models and [9, 42, 31, 26, 27, 29, 2] for stochastic volatility models and [35, 46] for model-free approaches.

Recently this asymptotic regime was also investigated for Asian options in [47] under the assumption that the asset price follows a local volatility model. The paper [47] considered arithmetic averaging Asian options in continuous time under the assumption that the asset price follows a local volatility model

$$(1) \quad dS_t = (r - q)S_t dt + \sigma(S_t)S_t dW_t, \quad S_0 > 0,$$

where W_t is a standard Brownian motion, $r \geq 0$ is the risk-free rate, $q \geq 0$ is the continuous dividend yield, $\sigma(\cdot)$ is the local volatility function. The local volatility function $\sigma(\cdot)$ was assumed to satisfy the boundedness and Lipschitz conditions

$$(2) \quad 0 < \underline{\sigma} \leq \sigma(\cdot) \leq \bar{\sigma} < \infty,$$

$$(3) \quad |\sigma(e^x) - \sigma(e^y)| \leq M|x - y|^\alpha,$$

for some fixed $M, \alpha > 0$ for any x, y and $0 < \underline{\sigma} < \bar{\sigma} < \infty$ are some fixed constants.

Under these assumptions, it is known from [51] that the log-stock price $X_t := \log S_t$ satisfies a sample path large deviation principle on an appropriate functional space. This result was used in [47] together with the contraction principle, to derive large deviations for the time average of the diffusion $\frac{1}{T} \int_0^T S_t dt$, and short maturity asymptotics for out-of-the-money (OTM) Asian options. Using call-put parity, the corresponding short maturity asymptotics for in-the-money (ITM) Asian options can be obtained as well. Finally, the short maturity asymptotics for at-the-money (ATM) Asian options has been derived too. The result in [47] covers in particular the Black-Scholes case, and the explicit formulas are derived for the short maturity asymptotics for the Black-Scholes case in [47].

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The assumptions (2), (3) are not satisfied by some of the models which are popular in financial practice. One important model of this type is the Constant Elasticity of Variance (CEV) model [15], which is defined by the diffusion

$$(4) \quad dS_t = (r - q)S_t dt + \sigma S_t^\beta dW_t, \quad S_0 > 0.$$

This model is used for modeling the skew in equities and FX markets, and allows the flexibility of calibrating to the ATM slope of the implied volatility by choosing appropriately the exponent β . For $\beta < 1$, the model reproduces the leverage effect observed in many financial markets, which is manifested as a decreasing volatility as the asset price increases. The result of this inverse relationship between the price and volatility is the implied volatility skew. See [44] for a survey of the mathematical properties of the CEV model and also the pricing of vanilla options under the CEV model.

In most practical applications the exponent β is usually chosen in the range $0 < \beta \leq 1$. The case $\beta = \frac{1}{2}$ corresponds to the square-root model of Cox and Ross [17], and is obtained as a particular case of the Feller process [25, 16]

$$(5) \quad dx_t = (bx_t + c)dt + \sqrt{2ax_t}dW_t,$$

with $a = \frac{1}{2}\sigma^2$, $b = r - q$, $c = 0$. The case of general β can also be mapped to the diffusion process (5) by a change of variable. The classification of the solutions of the process (5) has been studied by Feller [25]. This can be used to obtain the corresponding properties of the CEV model (4), which are summarized by the following well-known result, see [4, 44]:

(i) $0 < \beta < \frac{1}{2}$. The process (4) can be mapped to the diffusion (5) with $0 < c < a$. The fundamental solution of Fokker-Planck equation for the density of the diffusion (4) is not unique. There are two independent fundamental solutions, and the problem is well-posed only if we add an additional boundary condition at $x = 0$, for example absorbing or reflecting boundary condition.

(ii) $\frac{1}{2} \leq \beta < 1$. The process (4) can be mapped to the diffusion (5) with $c < 0$. The Fokker-Planck equation for the density of the diffusion has a unique fundamental solution, of decreasing norm.

The model (4) is a local volatility model of type (1) with a volatility function $\sigma(S_t) := \sigma S_t^{\beta-1}$. For $0 < \beta < 1$ this is not a bounded function. This implies the results of [51] cannot be directly applied to this case.

The pricing of Asian options has been widely studied in the mathematical finance literature. The pricing under the Black-Scholes model has been studied in [38, 11, 22, 43], using a relation between the distributional property of the time-integral of the geometric Brownian motion and Bessel processes. See [24] for an overview, and [33] for a comparison with alternative simulation methods, such as the Monte Carlo approach.

The PDE approach [48, 53, 54] can be used to price Asian options under a wide variety of models, using either a numerical approach [53, 54], or to derive analytical approximation formulae using asymptotic expansion methods. The paper [32] used heat kernel expansion methods and developed approximate formulae expressed in terms of elementary functions for the density, the price and the Greeks of path dependent options of Asian style. Asymptotic expansion leading to analytical approximations with error bounds for Asian options have been obtained also using Malliavin calculus in [49, 39].

Asian options pricing under the CEV model with $\beta = \frac{1}{2}$ has been studied in [23] and [18]. A detailed study under the $\beta = \frac{1}{2}$ model both with discrete and continuous time averaging was presented in [34]. The general case of the CEV model was studied in [32]

using heat kernel expansion methods in the PDE approach [48, 53, 54]. The paper [32] presented detailed numerical tests of their method under the CEV model, which show good convergence and stability of the expansion.

In this paper, we study the short maturity asymptotics for the price of the Asian options under the assumption that the underlying asset price follows the CEV model (4)¹. We consider both the fixed strike and floating strike Asian options. Our main tool is the large deviations theory from probability theory. For the theory and applications of large deviations, we refer to the book [19]. Some basic definitions and results needed in this paper will be provided in the Appendix.

The case of the square-root model $\beta = \frac{1}{2}$ is special as the model is affine, and the moment generating function of the time integral $\int_0^T S_t dt$ can be found in closed form. Then the application of the Gärtner-Ellis theorem gives the large deviations for the averaged time integral of the asset price.

For $\frac{1}{2} < \beta < 1$ we use a recent large deviations result due to Baldi and Caramelino [7] for the CEV model to derive a variational problem for the rate function determining the short maturity asymptotics of the Asian options. Large deviations for the square-root process $\beta = \frac{1}{2}$ were studied in [20]. The variational problem is solved completely. We derive large and small-strike asymptotics for the rate function.

Some of the methods proposed in the literature for pricing Asian options are less efficient in the small maturity/volatility limit. This is a well-known problem in many of the methods proposed for the Black-Scholes model [24], but a similar phenomenon appears also for the method of [18] in the square-root model, where the convergence of the expansion is slower for small maturity/volatility. The short maturity asymptotic expansion proposed in this paper complements the use of these methods in a regime where their numerical efficiency is less than optimal.

The paper is organized as follows. In Section 2, we present asymptotics for out-of-the-money (OTM) Asian options in the square-root model $\beta = \frac{1}{2}$. Section 3 considers the case of the general CEV model with $\frac{1}{2} \leq \beta < 1$. The asymptotics for OTM Asian options is given by the solution of a variational problem, which is solved in closed form. We also obtain the asymptotics for at-the-money (ATM) Asian options. Section 4 considers the asymptotics of Asian options with floating strike. In Section 5 we present an analytical approximation for the Asian options prices which has the same short maturity asymptotics as that obtained in Sections 2 and 3. This approximation is compared against benchmark results in the literature, and good agreement is demonstrated for model and option parameters relevant for practical applications. Finally, the background of large deviations theory and the proofs of the main results are given in the Appendix.

Notations and preliminaries. The price of the Asian call and put options with maturity T and strike K with continuous time averaging are given by expectations in the risk-neutral

¹We note that the short maturity asymptotics for vanilla options under the CEV model has been studied in the literature using several approaches, see [40, 42, 14].

measure

$$(6) \quad C(T) := e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right],$$

$$(7) \quad P(T) := e^{-rT} \mathbb{E} \left[\left(K - \frac{1}{T} \int_0^T S_t dt \right)^+ \right],$$

where $C(T)$ and $P(T)$ emphasize the dependence on the maturity T .

We denote the expectation of the averaged asset price in the risk-neutral measure as

$$(8) \quad A(T) := \frac{1}{T} \int_0^T \mathbb{E}[S_t] dt = S_0 \frac{1}{(r-q)T} \left(e^{(r-q)T} - 1 \right),$$

for $r - q \neq 0$ and $A(T) := S_0$ for $r - q = 0$. When $K > A(T)$, the call Asian option is out-of-the-money and $C(T) \rightarrow 0$ as $T \rightarrow 0$. When $A(T) > K$, the put Asian option is out-of-the-money and $P(T) \rightarrow 0$ as $T \rightarrow 0$.

The prices of call and put Asian options are related by put-call parity as

$$(9) \quad C(K, T) - P(K, T) = e^{-rT} (A(T) - K).$$

As $T \rightarrow 0$, we have $A(T) = S_0 + O(T)$. Therefore, for the small maturity regime, the call Asian option is out-of-the-money if and only if $K > S_0$ etc. For the purposes of the short maturity limit, the call Asian option is said to be out-of-the-money (resp. in-the-money) if $K > S_0$ (resp. $K < S_0$), and the put Asian option is said to be out-of-the-money (resp. in-the-money) if $K < S_0$ (resp. $K > S_0$), and finally they are said to be at-the-money if $K = S_0$.

2. SHORT MATURITY ASIAN OPTIONS IN THE SQUARE-ROOT MODEL

We assume in this Section that the asset value S_t follows a Square-root process:

$$(10) \quad dS_t = (r - q)S_t dt + \sigma \sqrt{S_t} dW_t,$$

with $S_0 > 0$ and W_t is a standard Brownian motion starting at zero at time zero $W_0 = 0$.

We have the following result.

Theorem 1. $\mathbb{P}(\frac{1}{T} \int_0^T S_t dt \in \cdot)$ satisfies a large deviation principle with rate function

$$(11) \quad \mathcal{I}(x, S_0) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda(\theta) \},$$

where

$$(12) \quad \Lambda(\theta) := \lim_{T \rightarrow 0} T \log \mathbb{E} \left[e^{\frac{\theta}{T^2} \int_0^T S_t dt} \right] = \begin{cases} \frac{\sqrt{2\theta}}{\sigma} \tan \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 & \text{if } 0 \leq \theta < \frac{\pi^2}{2\sigma^2} \\ \frac{-\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} \right) S_0 & \text{if } \theta \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

Indeed, the rate function in Theorem 1 has a more explicit expression. Together with Theorem 1 and Lemma 4 that we prove in Section 3, we have the following result.

Proposition 2. Assume the square root model: $\beta = \frac{1}{2}$.

(i) For $K \leq S_0$, the put option is OTM, and $P(T) = e^{-\frac{1}{T}\mathcal{I}(K, S_0) + o(1/T)}$, as $T \rightarrow 0$, where

$$(13) \quad \mathcal{I}(K, S_0) = \frac{S_0}{\sigma^2} \frac{x^2}{\cosh^2(x)} \left(\frac{\sinh(2x)}{2x} - 1 \right),$$

where x is the solution of the equation

$$(14) \quad \frac{1}{2 \cosh^2(x)} \left(1 + \frac{\sinh(2x)}{2x} \right) = \frac{K}{S_0}.$$

(ii) For $K \geq S_0$, the call option is OTM, and $C(T) = e^{-\frac{1}{T}\mathcal{I}(K, S_0) + o(1/T)}$, as $T \rightarrow 0$, where

$$(15) \quad \mathcal{I}(K, S_0) = \frac{S_0}{\sigma^2} \frac{x^2}{\cos^2(x)} \left(1 - \frac{\sin(2x)}{2x} \right),$$

where $0 \leq x \leq \frac{\pi}{2}$ is given by the solution of the equation

$$(16) \quad \frac{1}{2 \cos^2(x)} \left(1 + \frac{\sin(2x)}{2x} \right) = \frac{K}{S_0}.$$

We can study also the small/large strike asymptotics of the rate function.

Proposition 3. (i) The large strike asymptotics for the rate function of OTM Asian call options $K > S_0$ in the square-root model $\beta = \frac{1}{2}$ is

$$(17) \quad \lim_{K \rightarrow \infty} \frac{\mathcal{I}(K, S_0)}{K} = \frac{\pi^2}{2\sigma^2}.$$

(ii) The small strike $K \rightarrow 0$ asymptotics of the rate function for OTM Asian put options $K < S_0$ in the square-root model $\beta = \frac{1}{2}$ is

$$(18) \quad \mathcal{I}(K, S_0) \sim \frac{S_0^2}{2\sigma^2 K}, \quad \text{as } K \rightarrow 0.$$

2.1. Expansion of the rate function around the ATM point. We give also the expansion of the rate function for Asian options in the square-root model ($\beta = \frac{1}{2}$) in power series of $x = \log(K/S_0)$. The first few terms are

$$(19) \quad \mathcal{I}(K, S_0) = \frac{S_0}{\sigma^2} \left\{ \frac{3}{2}x^2 + \frac{3}{5}x^3 + \frac{271}{1400}x^4 + O(x^5) \right\}.$$

This gives an approximation for the rate function around the ATM point $x = 0$.

The rate function $\mathcal{I}(K, S_0)$ in the square-root model was evaluated numerically using the expression in Proposition 2. We show in Figure 1 the plot of this function vs. K/S_0 (left) and vs. $x = \log(K/S_0)$ (right). We show also in the right plot the approximation of the rate function obtained by keeping the first three terms in the series expansion (19), which gives a good approximation around the ATM point $x = 0$.

3. ASIAN OPTIONS IN THE CEV MODEL

The CEV model is defined by the one-dimensional diffusion under the risk-neutral measure

$$(20) \quad dS_t = (r - q)S_t dt + \sigma S_t^\beta dW_t,$$

with $S_0 > 0$.

It is easy to check that the following Lemma holds.

Lemma 4. *For an Asian OTM call option, that is, $K > S_0$, we have for $\frac{1}{2} \leq \beta < 1$*

$$(21) \quad \lim_{T \rightarrow 0} T \log C(T) = \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right).$$

For an Asian OTM put option, that is, $K \leq S_0$, we have for $\frac{1}{2} \leq \beta < 1$

$$(22) \quad \lim_{T \rightarrow 0} T \log P(T) = \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \leq K \right).$$

Using this result we can prove the short maturity asymptotics for OTM Asian options in the CEV model (4).

Theorem 5. *The short maturity asymptotics for OTM Asian options in the CEV model (4) with $\frac{1}{2} \leq \beta < 1$ is given by*

$$(23) \quad \lim_{T \rightarrow 0} T \log C(T) = -\mathcal{I}(K, S_0),$$

where the rate function is given by the solution of a variational problem specified as follows.

(i) *For OTM Asian call options $K > S_0$ we have*

$$(24) \quad \mathcal{I}(K, S_0) = \inf_{\int_0^1 g(t) dt \geq K, g(0)=S_0, g(t) \geq 0, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt, \quad K > S_0.$$

(ii) *For OTM Asian put options $K < S_0$ we have*

$$(25) \quad \mathcal{I}(K, S_0) = \inf_{\int_0^1 g(t) dt \leq K, g(0)=S_0, g(t) \geq 0, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt, \quad K < S_0.$$

3.1. At-the-Money Asian Options. Let us consider the ATM case, that is, $K = S_0 > 0$. For this case we have the following result.

Theorem 6. *As $T \rightarrow 0$, we have in the CEV model with $\frac{1}{2} \leq \beta < 1$*

$$(26) \quad C(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T), \quad P(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T).$$

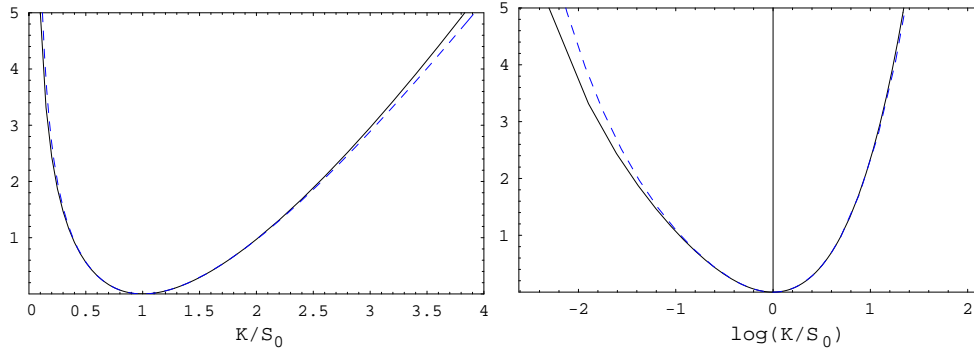


FIGURE 1. The rate function $\mathcal{I}(K, S_0)$ for $\beta = \frac{1}{2}$ in units of S_0/σ^2 vs K (left) and vs $\log(K/S_0)$ (right) (solid black curve) and the Taylor expansion (55) keeping the first three terms (dashed blue).

3.2. Variational Problem for Short-Maturity Asymptotics for Asian Options in the CEV model. Theorem 5 gives the rate function $\mathcal{I}(K, S_0)$ of an Asian option in the CEV model as a variational problem. For OTM Asian call option $K > S_0$ this variational problem reads

$$(27) \quad \mathcal{I}(K, S_0) = \inf_g \frac{1}{2\sigma^2} \int_0^1 \frac{(g'(t))^2}{g(t)^{2\beta}} dt,$$

where the function $g(t)$ is differentiable and satisfies $g(0) = S_0$, $g(t) > 0$, $0 \leq t \leq 1$ and the infimum is taken under the constraint

$$(28) \quad \int_0^1 g(t) dt \geq K.$$

Similarly, for OTM Asian put option with $K < S_0$, the rate function $\mathcal{I}(K, S_0)$ is given by the variational problem (27) with inequality constraint $\int_0^1 g(t) dt \leq K$.

Define $\mathcal{I}_K(K, S_0)$ as the solution of the variational problem (27), obtained by replacing the inequality (28) with equality. The strategy of the proof will be to show that $\mathcal{I}_K(K, S_0)$ is an increasing function for $K > S_0$ and thus the solution of the variational inequality is given by $\mathcal{I}(K, S_0) = \mathcal{I}_K(K, S_0)$. For $K < S_0$ we will show that $\mathcal{I}_K(K, S_0)$ is a decreasing function for $K < S_0$, and thus the solution of the variational inequality is given by $\mathcal{I}(K, S_0) = \mathcal{I}_K(K, S_0)$.

We give next the solution of the variational problem (27) with the equality constraint $\int_0^1 g(t) dt = K$. This is given by the following result.

Proposition 7. *The solution of the variational problem (27) with the equality constraint $\int_0^1 g(t) dt = K$ is given by*

$$(29) \quad \mathcal{I}_K(K, S_0) = \begin{cases} \frac{S_0^{2(1-\beta)}}{2\sigma^2} a^{(+)}(x) b^{(+)}(x) & K \leq S_0, \\ \frac{S_0^{2(1-\beta)}}{2\sigma^2} a^{(-)}(x) b^{(-)}(x) & K \geq S_0. \end{cases}$$

The two cases are as follows:

(i) $K \leq S_0$. $0 < x \leq 1$ is the solution of the equation

$$(30) \quad \frac{K}{S_0} = x + \frac{b^{(+)}(x)}{a^{(+)}(x)},$$

with

$$(31) \quad a^{(+)}(x) = 2x^{-\beta}(1-x)^{\frac{1}{2}} {}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right),$$

$$(32) \quad b^{(+)}(x) = \frac{2}{3}x^{-\beta}(1-x)^{\frac{3}{2}} {}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right).$$

The argument $z = 1 - \frac{1}{x}$ of the hypergeometric function ${}_2F_1(a, b; c; z)$ is negative.

(ii) $K \geq S_0$. $x \geq 1$ is the solution of the equation

$$(33) \quad \frac{K}{S_0} = x - \frac{b^{(-)}(x)}{a^{(-)}(x)},$$

with

$$(34) \quad a^{(-)}(x) = 2x^{-\beta}(x-1)^{\frac{1}{2}} {}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right),$$

$$(35) \quad b^{(-)}(x) = \frac{2}{3}x^{-\beta}(x-1)^{\frac{3}{2}} {}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right).$$

The argument $z = 1 - \frac{1}{x}$ of the hypergeometric function ${}_2F_1(a, b; c; z)$ is positive.

An alternative form of the solution for $\mathcal{I}_K(K, S_0)$ which gives additional information on the continuity and monotonicity properties of this function in K is given by the following result.

Proposition 8. (i) The function $\mathcal{I}_K(K, S_0)$ is given for $K > S_0$ by

$$(36) \quad \mathcal{I}_K(K, S_0) = \inf_{\varphi > K/S_0} \frac{1}{2} \frac{[\mathcal{G}^{(-)}(\varphi)]^2}{\varphi - \frac{K}{S_0}},$$

with

$$(37) \quad \begin{aligned} \mathcal{G}^{(-)}(\varphi) &= \frac{S_0^{1-\beta}}{\sigma} \int_1^\varphi z^{-\beta} \sqrt{\varphi - z} dz \\ &= \frac{S_0^{1-\beta}}{\sigma} \frac{2}{3} \varphi^{-\beta} (\varphi - 1)^{3/2} {}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\varphi}\right). \end{aligned}$$

(ii) The function $\mathcal{I}_K(K, S_0)$ is given for $K < S_0$ by

$$(38) \quad \mathcal{I}_K(K, S_0) = \inf_{0 < \chi < K/S_0} \frac{1}{2} \frac{[\mathcal{G}^{(+)}(\chi)]^2}{\frac{K}{S_0} - \chi},$$

with

$$(39) \quad \begin{aligned} \mathcal{G}^{(+)}(\chi) &= \frac{S_0^{1-\beta}}{\sigma} \int_\chi^1 z^{-\beta} \sqrt{z - \chi} dz \\ &= \frac{S_0^{1-\beta}}{\sigma} \frac{2}{3} \chi^{-\beta} (1 - \chi)^{3/2} {}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\chi}\right). \end{aligned}$$

From the representation of Proposition 8 it follows that $\mathcal{I}_K(K, S_0)$ is a continuous function of K . We also obtain the monotonicity properties of this function, which imply the relation to the rate function $\mathcal{I}(K, S_0)$ given by Theorem 5.

Corollary 9. We have the following monotonicity properties of the function $\mathcal{I}_K(K, S_0)$ with respect to strike K :

- (i) For $K > S_0$ the function $\mathcal{I}_K(K, S_0)$ is an increasing function of K .
- (ii) For $K < S_0$ the function $\mathcal{I}_K(K, S_0)$ is a decreasing function of K .
- (iii) The rate function $\mathcal{I}(K, S_0)$ is given by

$$(40) \quad \mathcal{I}(K, S_0) = \mathcal{I}_K(K, S_0).$$

Remark 10. For $\beta = \frac{1}{2}$, the results of Proposition 7 recover the result of Proposition 2. For this case the hypergeometric functions can be expressed in terms of elementary functions.

(i) For $K < S_0$ we need the expressions of $a^{(+)}(x), b^{(+)}(x)$ for x real and negative. We have for $z \in \mathbb{R}_+$

$$(41) \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z\right) = \frac{\operatorname{arcsinh}\sqrt{z}}{\sqrt{z}},$$

$$(42) \quad {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; -z\right) = -\frac{3}{2} \frac{\operatorname{arcsinh}\sqrt{z}}{z^{3/2}} + \frac{3}{2} \frac{\sqrt{1+z}}{z}.$$

The equation (30) reads

$$(43) \quad \frac{K}{S_0} = \frac{1}{2}x + \frac{1}{2} \frac{\sqrt{1-x}}{\operatorname{arcsinh}\frac{\sqrt{1-x}}{\sqrt{x}}}.$$

Denoting $x \rightarrow \frac{1}{\cosh^4 x}$ this equation becomes identical to (14).

The rate function (29) is

$$(44) \quad \mathcal{I}(K, S_0) = \frac{S_0}{\sigma^2} \left\{ -x \operatorname{arcsinh}^2 \frac{\sqrt{1-x}}{\sqrt{x}} + \sqrt{1-x} \operatorname{arcsinh} \frac{\sqrt{1-x}}{\sqrt{x}} \right\}.$$

Substituting here again $x \rightarrow \frac{1}{\cosh^4 x}$ this becomes identical with the result (13) for the rate function for $K < S_0$ in the square-root model.

(ii) A similar argument holds for $K > S_0$ using the expressions for the hypergeometric functions of positive argument

$$(45) \quad {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z\right) = \frac{\arcsin\sqrt{z}}{\sqrt{z}},$$

$$(46) \quad {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; z\right) = \frac{3}{2} \frac{\arcsin\sqrt{z}}{z^{3/2}} - \frac{3}{2} \frac{\sqrt{1-z}}{z}.$$

Remark 11. For $\beta \rightarrow 1$, the results of Proposition 7 recover the rate function for the Black-Scholes model in Proposition 12 of [47].

(i) For $K < S_0$ we need the expressions of $a^{(+)}(x), b^{(+)}(x)$ for $x \in (0, 1]$. We have for $z = \frac{1}{x} - 1 \in \mathbb{R}_+$

$$(47) \quad {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; -z\right) = \frac{\arctan\sqrt{z}}{\sqrt{z}},$$

$$(48) \quad {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; -z\right) = \frac{3}{z} - 3 \frac{\arctan\sqrt{z}}{z^{3/2}}.$$

The equation (30) reads

$$(49) \quad \frac{K}{S_0} = \frac{\sqrt{x(1-x)}}{\arctan\sqrt{1/x-1}}.$$

Identifying $\arctan\sqrt{1/x-1} = \xi$ this reproduces Eq. (33) in [47]

$$(50) \quad \frac{K}{S_0} = \frac{1}{2\xi} \sin(2\xi).$$

The rate function (29) becomes, when expressed in terms of ξ

$$(51) \quad \mathcal{I}(K, S_0) = \frac{1}{\sigma^2} 2\xi(\tan\xi - \xi),$$

which reproduces the result Eq. (31) in [47] for the rate function for $K < S_0$ in the BS model.

(ii) A similar argument holds for $K > S_0$ using the expressions for the hypergeometric functions of positive argument, $x \geq 1$

$$(52) \quad {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; z\right) = \frac{\operatorname{arctanh}\sqrt{z}}{\sqrt{z}}, \quad z = 1 - \frac{1}{x},$$

$$(53) \quad {}_2F_1\left(1, \frac{3}{2}; \frac{5}{2}; z\right) = -\frac{3}{z^{3/2}}(\sqrt{z} - \operatorname{arctanh}\sqrt{z}).$$

Identifying $\hat{\beta} = \frac{1}{2}\operatorname{arctanh}\sqrt{1 - \frac{1}{x}}$ we get the rate function

$$(54) \quad \mathcal{I}(K, S_0) = \frac{1}{\sigma^2} \left(\frac{1}{2}\hat{\beta}^2 - \hat{\beta} \tanh(\hat{\beta}/2) \right),$$

where $\hat{\beta}$ is the solution of the equation $\frac{K}{S_0} = \frac{1}{2\hat{\beta}} \sinh(2\hat{\beta})$. These are identical with the results of Proposition 12 of [47].

3.3. Expansion of the rate function around the ATM point. Using the same approach as in the proof of Proposition 14 in [47] one can expand the rate function in power series of $x = \log(K/S_0)$ for arbitrary β . The first few terms are

$$(55) \quad \mathcal{I}(K, S_0) = \frac{S_0^{2(1-\beta)}}{\sigma^2} \left\{ \frac{3}{2}x^2 + \left(-\frac{3}{10} + \frac{9}{5}(1-\beta) \right) x^3 + \left(\frac{109}{1400} - \frac{117}{350}(1-\beta) + \frac{198}{175}(1-\beta)^2 \right) x^4 + O(x^5) \right\}.$$

For $\beta = \frac{1}{2}$ this reduces to the expansion of the rate function in the square-root model given in equation (19).

3.4. Asymptotics of the rate function. We discuss next the asymptotics of the rate function $\mathcal{I}(K, S_0)$ in the CEV model for very small/large strike K . This is given by the following result, which generalizes the results of Proposition 3 to general $\frac{1}{2} \leq \beta < 1$.

Proposition 12 (Large strike asymptotics). *We have, for $\beta \in [\frac{1}{2}, 1)$,*

$$(56) \quad \mathcal{I}(K, S_0) \sim \frac{S_0^{2(1-\beta)}}{2\sigma^2} \frac{\pi\Gamma^2(1-\beta)}{(3-2\beta)\Gamma^2(3/2-\beta)} \left(\frac{3-2\beta}{2(1-\beta)} \frac{K}{S_0} \right)^{2(1-\beta)}, \quad \text{as } K \rightarrow \infty,$$

where $\Gamma(\cdot)$ is the Gamma function.

For $\beta = \frac{1}{2}$ this reproduces the result (i) of Proposition 3.

$$(57) \quad \mathcal{I}(K, S_0) \sim \frac{\pi^2 K}{2\sigma^2}, \quad \text{as } K \rightarrow \infty.$$

Proposition 13 (Small strike asymptotics). *The $K \rightarrow 0$ asymptotics of the rate function for $\beta \in (\frac{1}{2}, 1)$ is given by*

$$(58) \quad \lim_{K \rightarrow 0} \frac{K}{S_0} \mathcal{I}(K, S_0) = \frac{2S_0^{2(1-\beta)}}{\sigma^2(3-2\beta)^2}.$$

For $\beta \rightarrow \frac{1}{2}$, this reproduces the result (ii) of Proposition 3

$$(59) \quad \lim_{K \rightarrow 0} \frac{K}{S_0} \mathcal{I}(K, S_0) = \frac{2S_0^{2(1-\beta)}}{\sigma^2(3-2\beta)^2} \rightarrow \frac{S_0}{2\sigma^2},$$

4. FLOATING STRIKE ASIAN OPTIONS

We consider in this Section the short maturity asymptotics for floating strike Asian options. The prices of the floating strike Asian call/put options are given by risk-neutral expectations

$$(60) \quad C_f(T) := e^{-rT} \mathbb{E} \left[\left(\kappa S_T - \frac{1}{T} \int_0^T S_t dt \right)^+ \right],$$

$$(61) \quad P_f(T) := e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - \kappa S_T \right)^+ \right].$$

First of all, similar to Lemma 4, we have:

(i) For an Asian OTM call option, that is, $\kappa < 1$, we have for $\frac{1}{2} \leq \beta < 1$

$$(62) \quad \lim_{T \rightarrow 0} T \log C_f(T) = \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \leq \kappa S_T \right).$$

(ii) For an Asian OTM put option, that is, $\kappa > 1$, we have for $\frac{1}{2} \leq \beta < 1$

$$(63) \quad \lim_{T \rightarrow 0} T \log P_f(T) = \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq \kappa S_T \right).$$

We start by considering the square-root model:

$$(64) \quad dS_t = (r - q)S_t dt + \sigma \sqrt{S_t} dW_t,$$

with $S_0 > 0$ and W_t is a standard Brownian motion starting at zero at time zero.

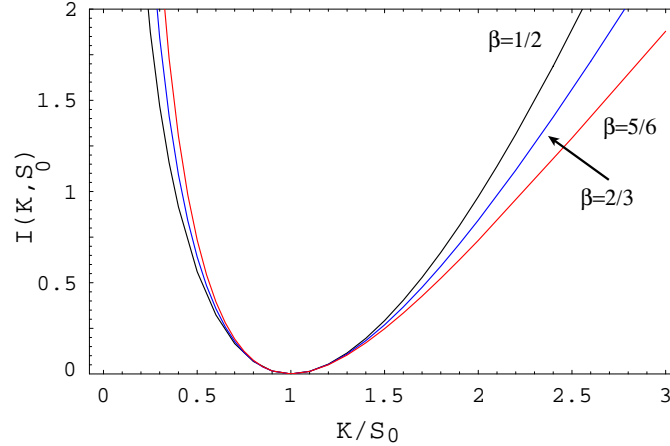


FIGURE 2. The rate function $\mathcal{I}(K, S_0) / (S_0^{2(1-\beta)} / \sigma^2)$ vs K/S_0 for Asian options in the CEV model with $\beta = \frac{1}{2}$ (black), $\beta = \frac{2}{3}$ (blue) and $\beta = \frac{5}{6}$ (red).

Theorem 14. For $\beta = \frac{1}{2}$, $\mathbb{P}\left(\frac{1}{T} \int_0^T S_t dt - \kappa S_T \in \cdot\right)$ satisfies a large deviation principle with the rate function

$$(65) \quad I_f(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda_f(\theta)\}$$

where $\Lambda_f(\theta)$ is given by

$$(66) \quad \Lambda_f(\theta) := \lim_{T \rightarrow 0} T \log \mathbb{E} \left[e^{\frac{\theta}{T^2} \int_0^T S_t dt} \right] \\ = \begin{cases} \frac{\sqrt{2\theta}}{\sigma} \tan \left(\frac{\sigma}{2} \sqrt{2\theta} + \tan^{-1} \left(-\sigma \kappa \sqrt{\frac{\theta}{2}} \right) \right) S_0 & \text{if } 0 \leq \theta < \theta_c \\ -\frac{\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} + \tanh^{-1} \left(-\sigma \kappa \sqrt{\frac{-\theta}{2}} \right) \right) S_0 & \text{if } \theta \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

where θ_c is the unique positive solution of the equation

$$(67) \quad \sqrt{\frac{\sigma^2 \theta_c}{2}} + \tan^{-1} \left(-\sigma \kappa \sqrt{\frac{\theta_c}{2}} \right) = \frac{\pi}{2}.$$

It follows from (62) and (63) that for $\kappa < 1$, the call option is OTM and $C_f(T) = e^{-\frac{1}{T} \mathcal{I}_f(\kappa, S_0) + o(1/T)}$, as $T \rightarrow 0$, and for $\kappa > 1$, the put option is OTM and $P_f(T) = e^{-\frac{1}{T} \mathcal{I}_f(\kappa, S_0) + o(1/T)}$, as $T \rightarrow 0$, where

$$(68) \quad \mathcal{I}_f(\kappa, S_0) = I_f(0) = \sup_{\theta \in \mathbb{R}} \{-\Lambda_f(\theta)\}.$$

The result of Theorem 14 for $\mathcal{I}_f(\kappa, S_0)$ for the square root model can be put into a more explicit form, as

$$(69) \quad \mathcal{I}_f(\kappa, S_0) = \frac{S_0}{\sigma^2} \mathcal{J}_f(\kappa),$$

where $\mathcal{J}_f(\kappa)$ is given by:

$$(70) \quad \text{(i) For } \kappa \geq 1 \quad \mathcal{J}_f(\kappa) = 2z \frac{\kappa z - \tan z}{1 + \kappa z \tan z},$$

where z is the solution of the equation

$$(71) \quad 1 + \kappa^2 z^2 + (1 - \kappa^2 z^2) \frac{\sin 2z}{2z} = 2\kappa \cos^2 z.$$

The solution is defined up to a sign, but this ambiguity is not relevant for computing $\mathcal{J}_f(\kappa)$.

(ii) For $\kappa \leq 1$

$$(72) \quad \mathcal{J}_f(\kappa) = 2z \frac{\kappa z - \tanh z}{1 - \kappa z \tanh z},$$

where z is the solution of the equation

$$(73) \quad 1 - \kappa^2 z^2 + (1 + \kappa^2 z^2) \frac{\sinh 2z}{2z} = 2\kappa \cosh^2 z.$$

The rate function $\mathcal{J}_f(\kappa, S_0)$ for the square root model $\beta = \frac{1}{2}$ is shown in Fig. 3 (solid black curve). This is compared against the rate function $\tilde{\mathcal{I}}(\kappa)$ for fixed strike Asian

options given by Proposition 2 (dashed curve). In the Black-Scholes model they are equal [47], which follows from the equivalence relations for fixed/floating strike Asian options [41]. These relations do not hold beyond the Black-Scholes model, and as a consequence the corresponding rate functions are different.

The floating strike rate function has the expansion around the ATM point $\kappa = 1$

$$(74) \quad \mathcal{J}_f(\kappa) = \frac{3}{2} \log^2 \kappa - \frac{33}{20} \log^3 \kappa + \frac{5809}{5600} \log^4 \kappa + O(\log^5 \kappa).$$

This is obtained by expanding the solution of the equations (71), (73) in powers of z , and inserting the result into (70) and (72). The approximation for the rate function $\mathcal{J}_f(\kappa)$ obtained by keeping the first three terms in this expansion is shown in Figure 3 as the solid blue curve.

For the general CEV model with $\frac{1}{2} \leq \beta < 1$, following the proof of Theorem 5, we get the following result:

Theorem 15. *The short maturity asymptotics for OTM floating strike Asian options in the CEV model (4) with $\frac{1}{2} \leq \beta < 1$ is given by*

(i) *For $\kappa < 1$, the short maturity asymptotics for OTM floating strike Asian call option is*

$$(75) \quad \lim_{T \rightarrow 0} T \log C_f(T) = -\mathcal{I}_f(\kappa, S_0),$$

where

$$(76) \quad \mathcal{I}_f(\kappa, S_0) = \inf_{\int_0^1 g(t) dt \leq \kappa g(1), g(0)=S_0, g(t) \geq 0, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt.$$

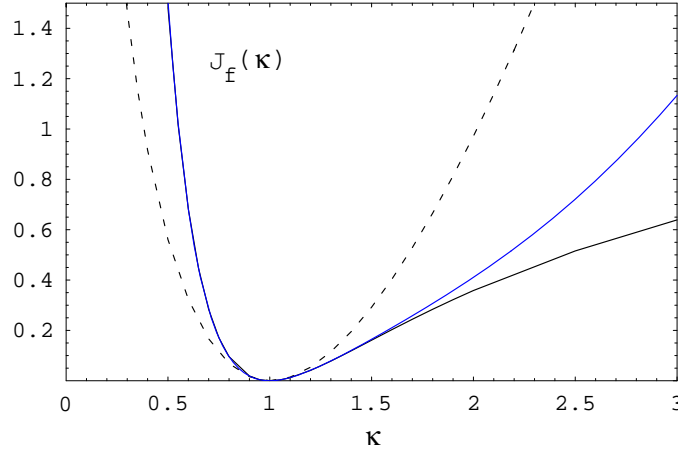


FIGURE 3. The rate function $\mathcal{J}_f(\kappa)$ vs κ for floating strike Asian options in the square root model with $\beta = \frac{1}{2}$ (black solid curve). The solid blue curve shows the approximation of this function obtained by keeping the first 3 terms in the expansion (74). This is compared against the fixed strike rate function $\mathcal{I}(S_0\kappa, S_0)$ (in units of S_0/σ^2) for the same model (dashed curve), given by Proposition 2.

(ii) For $\kappa > 1$, the short maturity asymptotics for OTM floating strike Asian put option is

$$(77) \quad \lim_{T \rightarrow 0} T \log P_f(T) = -\mathcal{I}_f(\kappa, S_0),$$

where

$$(78) \quad \mathcal{I}_f(\kappa, S_0) = \inf_{\int_0^1 g(t) dt \geq \kappa g(1), g(0)=S_0, g(t) \geq 0, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt.$$

Let us consider the ATM case, that is, $\kappa = 1$. For this case we have the following result. The proof is very similar to the proof of Theorem 6, and is hence omitted here.

Theorem 16. As $T \rightarrow 0$, we have in the CEV model with $\frac{1}{2} \leq \beta < 1$,

$$(79) \quad C_f(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T), \quad P_f(T) = \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{6\pi}} + O(T).$$

5. NUMERICAL TESTS

We present in this Section a few numerical tests of the short-maturity asymptotic results for Asian options in the CEV model obtained in this paper. Following [47] we will use the Asian option pricing formulas

$$(80) \quad C_{\text{asympt}}(K, T) = e^{-rT} (A(T)N(d_1) - KN(d_2)),$$

$$(81) \quad P_{\text{asympt}}(K, T) = e^{-rT} (KN(-d_2) - A(T)N(-d_1)),$$

where $A(T)$ is the expectation of the averaged asset price,

$$(82) \quad A(T) = S_0 \frac{1}{(r-q)T} (e^{(r-q)T} - 1),$$

and

$$(83) \quad d_{1,2} = \frac{1}{\Sigma_{LN} \sqrt{T}} \left(\log \frac{A(T)}{K} \pm \frac{1}{2} \Sigma_{LN}^2 T \right).$$

The equivalent log-normal volatility of the Asian option is defined by

$$(84) \quad \Sigma_{LN}^2(K, S_0) = \frac{\log^2(K/S_0)}{2\mathcal{I}(K, S_0)},$$

where $\mathcal{I}(K, S_0)$ is the rate function, given for the general CEV model in (29), and for the square-root model $\beta = \frac{1}{2}$ in Proposition 2. As shown in Proposition 18 of [47], the approximation (80),(81) has the same short maturity asymptotics as that given by Proposition 2 for the square-root model $\beta = \frac{1}{2}$, and by Theorem 5 for the general CEV model.

5.1. Equivalent log-normal volatility of Asian options in the CEV model. The series expansion of the equivalent log-normal volatility $\Sigma_{LN}(K, S_0)$ in powers of log-strike $x = \log(K/S_0)$ can be obtained by substituting (55) into the definition (84). This is

$$(85) \quad \Sigma_{LN}(K, S_0) = \sigma \frac{1}{\sqrt{3}} S_0^{\beta-1} \left\{ 1 + \left(\frac{1}{10} + \frac{3}{5}(\beta-1) \right) x + \left(-\frac{23}{2100} + \frac{12}{175}(\beta-1) + \frac{57}{350}(\beta-1)^2 \right) x^2 + O(x^3) \right\}.$$

For ATM Asian options $K = S_0$ the equivalent log-normal volatility is

$$(86) \quad \Sigma_{LN}(S_0, S_0) = \sigma \frac{1}{\sqrt{3}} S_0^{\beta-1}.$$

For the square-root model $\beta = \frac{1}{2}$ the ATM skew and convexity of the equivalent log-normal volatility are $-\frac{1}{5}$ and $-\frac{19}{4200}$ of the ATM equivalent volatility, respectively. We show in Figure 4 the plot of Σ_{LN}/σ vs $x = \log(K/S_0)$ obtained using equation (84) for the square-root model $\beta = \frac{1}{2}$.

The plot in Figure 4 is in general qualitative agreement with the shape of the implied volatility for options on realized variance in the Heston model, which are mathematically equivalent to Asian options in the square-root model. See Figure 5 (right) in [21]. As noted in the literature [21], the down-sloping shape of the implied volatility is a deficiency of the Heston model, as the observed smile for variance options in equity markets is up-sloping. The reference [21] proposes as an alternative model which has up-sloping smiles for variance options, the 3/2 stochastic volatility model [12, 13].

From the expansion (85) one can obtain the dependence of the ATM skew and convexity on the β parameter. For $\beta = \frac{1}{2}$ the ATM skew is negative; as β is increased, the ATM skew increases, crosses zero at $\beta = \frac{5}{6}$ and becomes positive. The ATM convexity is always negative for $\frac{1}{2} \leq \beta < 1$, so the equivalent log-normal volatility smile is slightly concave.

5.2. Numerical scenarios. We present next numerical tests for Asian option pricing in the square-root model $\beta = \frac{1}{2}$, for the 7 scenarios proposed in Dassios and Nagardjasarma [18]. We also compare with the third order approximation of Foschi, Pagliarani, Pascucci [32] (denoted as FPP3), listed in Table 5 of [32]. The results are shown in Table 1.

We note that the agreement of the asymptotic result with FPP3 is always better than 1% in relative value.

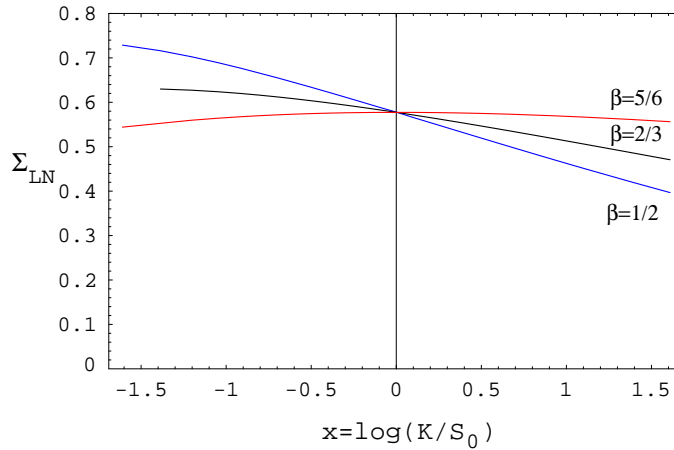


FIGURE 4. Small-maturity equivalent log-normal volatility $\Sigma_{LN}(K, S_0)/(\sigma S_0^{\beta-1})$ vs $x = \log(K/S_0)$ for an Asian option in the CEV model with $\beta = \frac{1}{2}$ (blue), $\beta = \frac{2}{3}$ (black) and $\beta = \frac{5}{6}$ (red).

TABLE 1. Comparison of the short-maturity asymptotic formulas for Asian options in the square-root model $\beta = \frac{1}{2}$ for the 7 scenarios considered by Dassios and Nagardjasarma [18]. The results are compared against those of [18] (DN) and those of Foschi, Pagliarani, Pascucci [32] (denoted as FPP3).

Case	S_0	K	r	σ	T	$C_{\text{asympt}}(K, T)$	DN	FPP3
1	2	2	0.01	0.14	1	0.055474	0.0197	0.055562
2	2	2	0.18	0.42	1	0.216013	0.2189	0.217874
3	2	2	0.0125	0.35	2	0.170568	0.1725	0.170926
4	1.9	2	0.05	0.69	1	0.189863	0.1902	0.190834
5	2	2	0.05	0.72	1	0.250113	NA	0.251121
6	2.1	2	0.05	0.72	1	0.307731	0.3098	0.308715
7	2	2	0.05	0.71	2	0.350516	0.3339	0.353197

TABLE 2. Numerical tests for the scenarios proposed in [18] and [32].

Case	σ	T	$C_{\text{asympt}}(K, T)$	DN	FPP3
1	0.71	0.1	0.075354	0.0751	0.075387
2	0.71	0.5	0.172813	0.1725	0.173175
3	0.71	1.0	0.247020	0.2468	0.248016
4	0.71	2.0	0.350516	0.3339	0.353197
5	0.71	5.0	0.536611	0.3733	0.545714
6	0.1	1.0	0.061310	0.0484	0.061439
7	0.3	1.0	0.120226	0.1207	0.120680
8	0.5	1.0	0.181983	0.1827	0.182723
9	0.7	1.0	0.243926	0.2446	0.244913

The agreement improves dramatically if we define the equivalent log-normal volatility as

$$(87) \quad \Sigma_{LN}^2(K, S_0) = \frac{\log^2(K/A(T))}{2\mathcal{I}(K, A(T))}.$$

However, the overall factor in $\mathcal{I}(K, S_0)$ must be still S_0 , not $A(T)$, which is somewhat arbitrary. Therefore we do not use this approximation. With this choice the agreement with FPP3 improves to better than 0.1% in relative value.

A second set of scenarios proposed by DN [18] is shown in Table 2. There are 9 scenarios with $S_0 = K = 2, r = 0.05, q = 0, \beta = \frac{1}{2}$. The asymptotic results are shown in Table 2, comparing with the results of [18] and [32] (Table 6 in this reference). Since they are all ATM scenarios, the use of the asymptotic formulas is very simple, and reduces to the use of equation (86).

The agreement of the asymptotic result with FPP3 is again very good, except for the $T = 5Y$ case. In all these cases (except $T = 5Y$) the difference between them is less than 1% in relative value. For maturities less than 1Y, the difference is always below 0.5% in relative value.

5.3. Floating-strike Asian options. We discuss also the pricing of floating-strike Asian options. They can be considered as call and put options on the underlying $B_T := \kappa S_T - A_T$. The forward price of this asset is

$$(88) \quad F_f(T) := \mathbb{E}[B_T] = S_0 \left(\kappa e^{(r-q)T} - \frac{e^{(r-q)T} - 1}{(r-q)T} \right).$$

For $\kappa \geq 0$, the underlying B_T takes values on the entire real axis. For this reason a Black-Scholes representation of this asset is not appropriate.

We propose to approximate the prices of floating-strike Asian options using a Bachelier (normal) approximation. These options are approximated as zero strike put and call options on the asset B_T , and their prices are

$$(89) \quad \begin{aligned} C_f(\kappa, T) &= e^{-rT} \left[F_f(T) \Phi(d) + \frac{1}{\sqrt{2\pi}} \Sigma_N \sqrt{T} e^{-\frac{1}{2}d^2} \right], \\ P_f(\kappa, T) &= e^{-rT} \left[-F_f(T) \Phi(-d) + \frac{1}{\sqrt{2\pi}} \Sigma_N \sqrt{T} e^{-\frac{1}{2}d^2} \right], \end{aligned}$$

with $d = \frac{F_f(T)}{\Sigma_N \sqrt{T}}$.

The equivalent normal volatility $\Sigma_N(\kappa, T)$ is specified by requiring that the small-maturity asymptotics of the floating-strike Asian options matches that of the Bachelier expression. This is given by the following result.

Proposition 17. *The short-maturity limit of the equivalent normal volatility in the square-root model $\beta = \frac{1}{2}$ is given by:*

(i) *for OTM floating strike Asian options $\kappa \neq 1$*

$$(90) \quad \lim_{T \rightarrow 0} \Sigma_N(\kappa, T) = \frac{\sigma^2}{2S_0} \frac{(\kappa - 1)^2}{\mathcal{J}_f(\kappa)},$$

where $\mathcal{J}_f(\kappa)$ is given by (69).

(ii) *for ATM floating strike Asian options $\kappa = 1$*

$$(91) \quad \lim_{T \rightarrow 0} \Sigma_N(\kappa, T) = \sigma \sqrt{\frac{S_0}{3}}.$$

Proof. The proof is similar to that of Proposition 18 in [47] and will be omitted. \square

The pricing of floating-strike Asian options in the square-root model has been considered in [34]. This paper studied the pricing of options with payoff $(-S_T + A_T - K)^+$ with K both positive and negative, using both discrete and continuous time monitoring. We will compare the result for $K = 0$ with continuous time averaging, which corresponds in our notations to a floating strike Asian put option with $\kappa = 1$.

The model parameters used in [34] are $S_0 = 1, r = 0.04, \sigma = 0.7$, and the option maturity is $T = 1$. The price quoted in Table 3 of this paper with $K = 0$ is $C_f(1, T) = 0.14376$. The asymptotic formula (89) gives $C_f(1, T) = 0.14524$, which is in reasonably good agreement with the result of [34] (1% relative difference).

6. APPENDIX: PROOFS

6.1. Background of Large Deviations Theory. We start by giving a formal definition of the large deviation principle. We refer to Dembo and Zeitouni [19] for general background of large deviations theory and its applications.

Definition 18 (Large Deviation Principle). *A sequence $(P_\epsilon)_{\epsilon \in \mathbb{R}^+}$ of probability measures on a topological space X satisfies the large deviation principle with rate function $I : X \rightarrow \mathbb{R}$ if I is non-negative, lower semicontinuous and for any measurable set A , we have*

$$(92) \quad - \inf_{x \in A^\circ} I(x) \leq \liminf_{\epsilon \rightarrow 0} \epsilon \log P_\epsilon(A) \leq \limsup_{\epsilon \rightarrow 0} \epsilon \log P_\epsilon(A) \leq - \inf_{x \in \bar{A}} I(x).$$

Here, A° is the interior of A and \bar{A} is its closure.

The contraction principle plays a key role in our proofs. For the convenience of the readers, we state the result as follows:

Theorem 19 (Contraction Principle, e.g. Theorem 4.2.1. [19]). *If P_ϵ satisfies a large deviation principle on X with rate function $I(x)$ and $F : X \rightarrow Y$ is a continuous map, then the probability measures $Q_\epsilon := P_\epsilon F^{-1}$ satisfies a large deviation principle on Y with rate function*

$$(93) \quad J(y) = \inf_{x: F(x)=y} I(x).$$

We will use the following version of the Gärtner-Ellis Theorem in the proofs in this paper.

Theorem 20 (Gärtner-Ellis Theorem, e.g. Theorem [19]). *Let Z_ϵ be a sequence of random variables on \mathbb{R} . Assume the limit $\Lambda(\theta) := \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{\frac{\theta}{\epsilon} Z_\epsilon}]$ exists on the extended real line and the interior of the set $\mathcal{D} := \{\theta : \Lambda(\theta) < \infty\}$ contains 0, and $\Lambda(\theta)$ is differentiable for any θ in the interior of \mathcal{D} and $|\Lambda'(\theta)| \rightarrow \infty$ as θ approaches to the boundary of \mathcal{D} . Then $\mathbb{P}(Z_\epsilon \in \cdot)$ satisfies a large deviation principle with the rate function $I(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda(\theta)\}$.*

6.2. Proofs of the results in Section 2.

Proof of Theorem 1. For any $\theta \in \mathbb{R}$, $u(t, x) = \mathbb{E}[e^{\theta \int_0^t S_s ds} | S_0 = x]$ satisfies the PDE:

$$(94) \quad \frac{\partial u}{\partial t} = (r - q)x \frac{\partial u}{\partial x} + \frac{1}{2} \sigma^2 x \frac{\partial^2 u}{\partial x^2} + \theta x u(t, x),$$

with $u(0, x) \equiv 1$. This affine PDE has the solution $u(t, x) = e^{A(t)x + B(t)}$, where

$$(95) \quad A'(t) = (r - q)A(t) + \frac{1}{2} \sigma^2 A(t)^2 + \theta,$$

$$(96) \quad B'(t) = 0,$$

with $A(0) = B(0) = 0$ and hence $B(t) = 0$ and for $\theta > 0$ sufficiently large,

$$(97) \quad \frac{2}{\sqrt{2\sigma^2\theta - (r - q)^2}} \tan^{-1} \left(\frac{r - q + \sigma^2 A}{\sqrt{2\sigma^2\theta - (r - q)^2}} \right) \Big|_{A=0}^{A=A(t)} = t,$$

and thus

$$(98) \quad A(t; \theta) = \frac{\sqrt{2\sigma^2\theta - (r-q)^2}}{\sigma^2} \tan \left[\frac{\sqrt{2\sigma^2\theta - (r-q)^2}}{2} t + \tan^{-1} \left(\frac{r-q}{\sqrt{2\sigma^2\theta - (r-q)^2}} \right) \right] - \frac{r-q}{\sigma^2}.$$

For $\theta < 0$ sufficiently negative,

$$(99) \quad \frac{2}{\sigma^2} \frac{1}{2\sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}}} \log \left(\frac{\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + A}{\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + A} \right) \Big|_{A=0}^{A=A(t)} = t,$$

and thus

$$(100) \quad A(t; \theta) = \frac{e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}} - 1}{\frac{1}{\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}}} - \frac{e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}}{\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}}}} = \frac{\frac{2\theta}{\sigma^2} (e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}} - 1)}{\frac{r-q}{\sigma^2} (1 - e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}) + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} (e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}} + 1)}.$$

Let us study now the $T \rightarrow 0$ limit. We note that for any $T > 0$ sufficiently small, we have

$$(101) \quad \mathbb{E} \left[e^{\frac{\theta}{T^2} \int_0^T S_t dt} \right] = e^{A(T; \frac{\theta}{T^2}) S_0}.$$

For $0 \leq \theta < \frac{\pi^2}{2\sigma^2}$,

$$(102) \quad \lim_{T \rightarrow 0} T A \left(T; \frac{\theta}{T^2} \right) = \sqrt{\frac{2\theta}{\sigma^2}} \tan \sqrt{\frac{\sigma^2\theta}{2}},$$

and this limit is ∞ if $\theta \geq \frac{\pi^2}{2\sigma^2}$.

For $\theta < 0$,

$$(103) \quad \lim_{T \rightarrow 0} T A \left(T; \frac{\theta}{T^2} \right) = \frac{-\sqrt{-2\theta}}{\sigma} \frac{e^{\sigma\sqrt{-2\theta}} - 1}{e^{\sigma\sqrt{-2\theta}} + 1} = \frac{-\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} \right).$$

Therefore,

$$(104) \quad \Lambda(\theta) := \lim_{T \rightarrow 0} T \log \mathbb{E} \left[e^{\frac{\theta}{T^2} \int_0^T S_t dt} \right] = \begin{cases} \frac{\sqrt{2\theta}}{\sigma} \tan \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 & \text{if } 0 \leq \theta < \frac{\pi^2}{2\sigma^2} \\ \frac{-\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} \right) S_0 & \text{if } \theta \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

For $0 < \theta < \frac{\pi^2}{2\sigma^2}$ and $\theta < 0$, $\Lambda(\theta)$ is differentiable and it is also easy to check that $\Lambda(\theta)$ is differentiable at $\theta = 0$. Finally, for $0 < \theta < \frac{\pi^2}{2\sigma^2}$, we can compute that

$$(105) \quad \frac{\partial \Lambda(\theta)}{\partial \theta} = \frac{\sqrt{2}}{\sigma 2\sqrt{\theta}} \tan \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 + \frac{\sqrt{2\theta}}{\sigma} \frac{\sigma\sqrt{2}}{4\sqrt{\theta}} \sec^2 \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 \rightarrow +\infty,$$

as $\theta \uparrow \frac{\pi^2}{2\sigma^2}$. Hence, we proved the essential smoothness condition. The conclusion follows from the Gärtner-Ellis theorem, see Theorem 20 in the Appendix. \square

Proof for Proposition 2. The result follows from the Theorem 1 and the Gärtner-Ellis theorem. According to this result the rate function is given by the Legendre transform of the cumulant function

$$(106) \quad \mathcal{I}(K, S_0) = \sup_{\theta \in \mathbb{R}} \{\theta K - \Lambda(\theta)\},$$

where the cumulant function $\Lambda(\theta)$ is given by Theorem 1.

(i) $K \geq S_0$. This case corresponds to $0 \leq \theta \leq \frac{\pi^2}{2\sigma^2}$. The cumulant function $\Lambda(\theta)$ is given by

$$(107) \quad \Lambda(\theta) = \frac{S_0}{\sigma^2} \sqrt{2\theta\sigma^2} \tan \sqrt{\frac{1}{2}\sigma^2\theta} = \frac{S_0}{\sigma^2} F_+(\theta\sigma^2).$$

where we defined $F_+(y) := \sqrt{2y} \tan \sqrt{\frac{1}{2}y}$.

The optimal value of θ in (106) is given by the solution of the equation

$$(108) \quad K = S_0 F'_+(\theta_*\sigma^2),$$

with

$$(109) \quad F'_+(y) = \frac{1}{2 \cos^2 \sqrt{y/2}} \left(1 + \frac{\sin \sqrt{2y}}{\sqrt{2y}} \right).$$

Numerical evaluation shows that $F'_+(y) : [0, \infty) \rightarrow [1, \infty)$ is a bijective map, such that this equation will have a solution for $K > S_0$. Identifying

$$(110) \quad x = \sqrt{\frac{1}{2}\theta_*\sigma^2},$$

it is easy to see that the equation for θ_* is the same as (16). The result for the rate function is

$$(111) \quad \begin{aligned} \mathcal{I}(K, S_0) &= \theta_* K - \Lambda(\theta_*) = \frac{S_0}{\sigma^2} \left(\theta_* \sigma^2 \frac{K}{S_0} - F_+(\theta_* \sigma^2) \right) \\ &= \frac{S_0}{\sigma^2} \left(2x^2 \frac{1}{2 \cos^2 x} \left(1 + \frac{\sin 2x}{2x} \right) - 2x \tan x \right) \\ &= \frac{S_0}{\sigma^2} \frac{x^2}{\cos^2 x} \left(1 - \frac{\sin(2x)}{2x} \right), \end{aligned}$$

which yields equation (15).

(ii) $K \leq S_0$. This case corresponds to $\theta \leq 0$. The cumulant function $\Lambda(\theta)$ is

$$(112) \quad \Lambda(\theta) = -\frac{S_0}{\sigma^2} \sqrt{-2\theta\sigma^2} \tanh \sqrt{-\frac{1}{2}\theta\sigma^2} = \frac{S_0}{\sigma^2} F_-(\theta\sigma^2),$$

where we introduced $F_-(y) := -\sqrt{-2y} \tanh \sqrt{-\frac{1}{2}y}$. This is related to the function appearing for the previous case as $F_-(iy) = F_+(y)$.

The optimal θ is given by the solution of the equation

$$(113) \quad \frac{K}{S_0} = F'_-(\theta_*\sigma^2),$$

where

$$(114) \quad F'_-(y) = \frac{1}{2 \cosh^2 \sqrt{-\frac{1}{2}y}} \left(1 + \frac{\sinh \sqrt{-2y}}{\sqrt{-2y}} \right).$$

Numerical evaluation gives that $F'_-(y) : (-\infty, 0] \rightarrow (0, 1]$ is a bijective function, so this equation will have a solution for $K < S_0$. Identifying

$$(115) \quad x = \sqrt{-\frac{1}{2}\theta_*\sigma^2}.$$

we see that the equation (113) reproduces (14). The result for the rate function is

$$(116) \quad \begin{aligned} \mathcal{I}(K, S_0) &= \theta_* K - \Lambda(\theta_*) = \frac{S_0}{\sigma^2} \left(\theta_* \sigma^2 \frac{K}{S_0} - F_-(\theta_* \sigma^2) \right) \\ &= \frac{S_0}{\sigma^2} \left(-2x^2 \frac{1}{2 \cosh^2 x} \left(1 + \frac{\sinh 2x}{2x} \right) + 2x \tanh x \right) \\ &= -\frac{S_0}{\sigma^2} \frac{x^2}{\cosh^2 x} \left(1 - \frac{\sinh(2x)}{2x} \right), \end{aligned}$$

which gives the result of equation (13). □

Proof of Proposition 3. (i) This is obtained starting with the relation

$$(117) \quad \mathcal{I}(K, S_0) = \sup_{0 \leq \theta < \frac{\pi^2}{2\sigma^2}} \left\{ \theta K - \frac{\sqrt{2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 \right\}.$$

On the one hand, $\mathcal{I}(K, S_0) \leq \sup_{0 \leq \theta < \frac{\pi^2}{2\sigma^2}} \theta K = \frac{\pi^2}{2\sigma^2} K$. On the other hand, for any $\epsilon > 0$, for sufficiently large K ,

$$(118) \quad \mathcal{I}(K, S_0) = \sup_{\frac{\pi^2}{2\sigma^2} - \epsilon \leq \theta < \frac{\pi^2}{2\sigma^2}} \left\{ \theta K - \frac{\sqrt{2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{2\theta} \right) S_0 \right\} \geq \left(\frac{\pi^2}{2\sigma^2} - \epsilon \right) K - \Lambda \left(\frac{\pi^2}{2\sigma^2} - \epsilon \right).$$

Thus, $\liminf_{K \rightarrow \infty} \frac{\mathcal{I}(K, S_0)}{K} \geq \left(\frac{\pi^2}{2\sigma^2} - \epsilon \right)$. Since it holds for any $\epsilon > 0$, we conclude that the relation (17) holds.

(ii) This is obtained starting from the relation

$$(119) \quad \mathcal{I}(K, S_0) = \sup_{\theta \leq 0} \left\{ K\theta + \frac{\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} \right) S_0 \right\}.$$

At optimality we have

$$(120) \quad K = \frac{\sqrt{2}}{2\sigma\sqrt{-\theta}} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} \right) S_0 + \frac{1}{2} \left[1 - \tanh^2 \left(\frac{\sigma}{2} \sqrt{-2\theta} \right) \right] S_0.$$

Note that the function $\tanh x$ approaches 1 exponentially fast as $x \rightarrow \infty$. Therefore, $\theta \sim -\frac{S_0^2}{2\sigma^2 K^2}$ as $K \rightarrow 0$ and the result (18) follows. □

6.3. Proofs of the results in Section 3.

Proof of Lemma 4. We will prove the result for the case of the Asian call option. The case of the Asian put option is very similar.

Note that by Hölder's inequality, for any $\frac{1}{p} + \frac{1}{p'} = 1$, $p, p' > 1$ and $p \geq 2$,

$$\begin{aligned}
 (121) \quad C(T) &= e^{-rT} \mathbb{E} \left[\left| \frac{1}{T} \int_0^T S_t dt - K \right| 1_{\frac{1}{T} \int_0^T S_t dt \geq K} \right] \\
 &\leq e^{-rT} \left(\mathbb{E} \left[\left| \frac{1}{T} \int_0^T S_t dt - K \right|^p \right] \right)^{\frac{1}{p}} \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right)^{\frac{1}{p'}} \\
 &\leq e^{-rT} 2^{\frac{p-1}{p}} \left(K^p + \left(\mathbb{E} \left[\frac{1}{T} \int_0^T S_t^p dt \right] \right)^{1/p} \right)^{1/p} \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right)^{\frac{1}{p'}},
 \end{aligned}$$

where in the last step we used Jensen's inequality to write

$$\begin{aligned}
 (122) \quad \mathbb{E} \left[\left| \frac{1}{T} \int_0^T S_t dt - K \right|^p \right] &\leq \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt + K \right)^p \right] \\
 &\leq 2^{p-1} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt \right)^p + K^p \right] \leq 2^{p-1} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t^p dt \right) + K^p \right].
 \end{aligned}$$

The second inequality follows by noting that for $p \geq 2$, $x \rightarrow x^p$ is a convex function for $x \geq 0$, which gives by Jensen's inequality $\left(\frac{x+y}{2}\right)^p \leq \frac{x^p+y^p}{2}$ for any $x, y \geq 0$. This gives

$$\begin{aligned}
 (123) \quad \mathbb{E} \left[\left| \frac{1}{T} \int_0^T S_t dt - K \right|^p \right] &\leq \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt + K \right)^p \right] \\
 &\leq 2^{p-1} \left[\mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt \right)^p \right] + K^p \right].
 \end{aligned}$$

The last inequality follows again from the Jensen's inequality which gives for $p \geq 2$ $\mathbb{E}[(\frac{1}{T} \int_0^T S_t dt)^p] \leq \mathbb{E}[\frac{1}{T} \int_0^T S_t^p dt]$.

For any $p \geq 2$,

$$(124) \quad \frac{1}{T} \int_0^T \mathbb{E}[S_t^p] dt = O(1),$$

since for the CEV process, all these moments are finite and well-behaved as $T \rightarrow 0$. The marginal distribution of S_t in this model is known [44] and the above expression can be computed explicitly.

Therefore, we have

$$(125) \quad \limsup_{T \rightarrow 0} T \log C(T) \leq \limsup_{T \rightarrow 0} \frac{1}{p'} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right).$$

Since it holds for any $2 > p' > 1$, we have the upper bound.

Next we derive a matching lower bound on $C(T)$. For any $\epsilon > 0$,

$$(126) \quad \begin{aligned} C(T) &\geq e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right) 1_{\frac{1}{T} \int_0^T S_t dt \geq K + \epsilon} \right] \\ &\geq e^{-rT} \epsilon \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K + \epsilon \right), \end{aligned}$$

which implies that

$$(127) \quad \liminf_{T \rightarrow 0} T \log C(T) \geq \liminf_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K + \epsilon \right).$$

Since it holds for any $\epsilon > 0$, we get the lower bound by letting $\epsilon \rightarrow 0$, provided that the limit $\mathcal{I}(K, S_0) := -\lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right)$ exists and is continuous in K . The continuity in K can be seen from the expression in Proposition 8. \square

Proof of Theorem 5. We split the proof into several steps.

Step 1. We need to prove that

$$(128) \quad \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right) = \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T \hat{S}_t dt \geq K \right),$$

where

$$(129) \quad d\hat{S}_t = \sigma \hat{S}_t^\beta dW_t,$$

with $\hat{S}_0 = S_0$. That is, the drift term is negligible for small time large deviations. Let us now prove (128). Note that

$$(130) \quad S_t = S_0 e^{(r-q)t + \int_0^t \sigma S_s^\beta dW_s - \frac{1}{2} \sigma^2 \int_0^t S_s^{2\beta} ds} = e^{(r-q)t} \tilde{S}_t,$$

where

$$(131) \quad d\tilde{S}_t = \sigma \tilde{S}_t^\beta e^{-(r-q)\beta t} dW_t, \quad \tilde{S}_0 = S_0 > 0.$$

By the time change $d\tau(t) = e^{-2(r-q)\beta t} dt$, $\tau(0) = 0$, $\tilde{S}_t = \hat{S}_{\tau(t)}$, where \hat{S} is defined in (129).

Hence,

$$(132) \quad \begin{aligned} &\lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt \geq K \right) \\ &= \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^T e^{(r-q)t} \hat{S}_{\tau(t)} dt \geq K \right) \\ &= \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\frac{1}{T} \int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)} \hat{S}_t dt \geq K \right). \end{aligned}$$

It is easy to check that $\frac{\tau(T)}{T} \rightarrow 1$ as $T \rightarrow 0$ and $\lim_{T \rightarrow 0} \inf_{0 \leq t \leq T} e^{(r-q)(1-2\beta)\tau^{-1}(t)} = \lim_{T \rightarrow 0} \sup_{0 \leq t \leq T} e^{(r-q)(1-2\beta)\tau^{-1}(t)} = 1$. Hence, (128) follows.

Step 2. Now assume that $r = q = 0$ so that

$$(133) \quad dS_t = \sigma S_t^\beta dW_t,$$

with $S_0 > 0$. Therefore, for $0 \leq t \leq 1$,

$$(134) \quad dS_{tT} = \sigma S_{tT}^\beta dW_{tT} = \sqrt{T} \sigma S_{tT}^\beta d(W_{tT}/\sqrt{T}) = \sqrt{T} \sigma S_{tT}^\beta dB_t,$$

where $B_t := W_{tT}/\sqrt{T}$ is a standard Brownian motion by the scaling property of the Brownian motion. Therefore, by letting $T = \epsilon$,

$$(135) \quad \lim_{T \rightarrow 0} T \log \mathbb{P} \left(\int_0^1 S_{tT} dt \geq K \right) = \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right),$$

where

$$(136) \quad dS_t^\epsilon = \sqrt{\epsilon} \sigma(S_t^\epsilon)^\beta dB_t,$$

with $S_0^\epsilon = S_0 > 0$.

Step 3. We need to show that

$$(137) \quad \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right) = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right).$$

Note that conditional of $\int_0^1 S_t^\epsilon dt \geq K$, the event that $S_t^\epsilon \geq \delta, 0 \leq t \leq 1$ is a typical event, while the event that $S_t^\epsilon \leq \delta$ for some $0 \leq t \leq 1$ is a rare event. Therefore, for sufficiently small $\delta > 0$,

$$(138) \quad \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right) \leq 2 \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right).$$

On the other hand, for any $\delta > 0$,

$$(139) \quad \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right) \geq \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right),$$

which implies that, for any $\delta > 0$.

$$(140) \quad \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right) \geq \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right).$$

Hence, (137) follows from (138) and (140).

Step 4. Define

$$(141) \quad dS_t^{\epsilon, \delta} = b^\delta(S_t^{\epsilon, \delta}) dt + \sqrt{\epsilon} \sigma(S_t^{\epsilon, \delta})^\beta dB_t, \quad S_0^{\epsilon, \delta} = S_0,$$

where $b^\delta(x) = 0$ for any $x > \delta$ and also is locally Lipschitz continuous and $b^\delta(0) > 0$. Moreover, $S \mapsto S^\beta$ is Hölder continuous with exponent $\geq \frac{1}{2}$ and for $\beta < 1$, it has sublinear growth at ∞ . The dynamics (141) satisfies the assumption A1.1. in Baldi and Caramellino [7]. It is easy to see that

$$(142) \quad \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right) = \mathbb{P} \left(\int_0^1 S_t^{\epsilon, \delta} dt \geq K, S_t^{\epsilon, \delta} \geq \delta, 0 \leq t \leq 1 \right).$$

By Theorem 1.2 in Baldi and Caramellino [7] it follows that $\mathbb{P}(S^{\epsilon, \delta} \in \cdot)$ satisfies a large deviation principle on $C_{S_0}([0, 1])$, the space of continuous functions starting at S_0 equipped with uniform topology, with the rate function $\frac{1}{2} \int_0^1 \frac{(g'(t) - b^\delta(g(t)))^2}{\sigma^2 g(t)^{2\beta}} dt$, with the understanding that the rate function is $+\infty$ if g is not differentiable. Moreover, the map $g \mapsto (\int_0^1 g(t) dt, g)$ is continuous from $C_{S_0}[0, 1]$ to $\mathbb{R}_+ \times C_{S_0}[0, 1]$.

By the contraction principle, see Theorem 19 in the Appendix, we have

$$\begin{aligned}
 (143) \quad & \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^{\epsilon, \delta} \geq \delta, 0 \leq t \leq 1 \right) \\
 &= - \inf_{\int_0^1 g(t) dt \geq K, g(0)=S_0, g(t) \geq \delta, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t) - b^\delta(g(t)))^2}{\sigma^2 g(t)^{2\beta}} dt \\
 &= - \inf_{\int_0^1 g(t) dt \geq K, g(0)=S_0, g(t) \geq \delta, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (144) \quad & \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K \right) \\
 &= \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P} \left(\int_0^1 S_t^\epsilon dt \geq K, S_t^\epsilon \geq \delta, 0 \leq t \leq 1 \right) \\
 &= - \inf_{\int_0^1 g(t) dt \geq K, g(0)=S_0, g(t) \geq 0, 0 \leq t \leq 1} \frac{1}{2} \int_0^1 \frac{(g'(t))^2}{\sigma^2 g(t)^{2\beta}} dt.
 \end{aligned}$$

□

Proof of Theorem 6. We will only prove the case for the call option here. The proof for the put option is very similar and hence omitted. As $T \rightarrow 0$,

$$(145) \quad C(T) = e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] = \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] + O(T),$$

and we showed that

$$(146) \quad \mathbb{E} \left[\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ \right] = \mathbb{E} \left[\left(\frac{1}{T} \int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)} \hat{S}_t dt - K \right)^+ \right],$$

where $d\hat{S}_t = \sigma \hat{S}_t^\beta dW_t$ and $\hat{S}_0 = S_0$.

It is easy to show that

$$\begin{aligned}
 (147) \quad & \left| \mathbb{E} \left[\left(\frac{1}{T} \int_0^{\tau(T)} e^{(r-q)(1-2\beta)\tau^{-1}(t)} \hat{S}_t dt - K \right)^+ \right] - \mathbb{E} \left[\left(\frac{1}{T} \int_0^{\tau(T)} \hat{S}_t dt - K \right)^+ \right] \right| \\
 &\leq \mathbb{E} \left[\frac{1}{T} \int_0^{\tau(T)} |e^{(r-q)(1-2\beta)\tau^{-1}(t)} - 1| \hat{S}_t dt \right] \\
 &= S_0 \frac{1}{T} \int_0^{\tau(T)} |e^{(r-q)(1-2\beta)\tau^{-1}(t)} - 1| dt = O(T).
 \end{aligned}$$

Moreover, we can show that

$$\begin{aligned}
 (148) \quad & \left| \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \hat{S}_t dt - K \right)^+ \right] - \mathbb{E} \left[\left(\frac{1}{T} \int_0^{\tau(T)} \hat{S}_t dt - K \right)^+ \right] \right| \\
 &\leq \mathbb{E} \left| \frac{1}{T} \int_{\tau(T)}^T \hat{S}_t dt \right| = S_0 \frac{1}{T} |T - \tau(T)| = O(T).
 \end{aligned}$$

Next, let $dX_t = \sigma S_0^\beta dW_t$ and $X_0 = S_0$, that is $X_t = S_0 + \sigma S_0^\beta W_t$. By Itô's formula and taking the expectations, we get

$$(149) \quad \begin{aligned} \mathbb{E}(\hat{S}_t - X_t)^2 &= \sigma^2 \int_0^t \mathbb{E}(\hat{S}_s^\beta - S_0^\beta)^2 ds \\ &\leq 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2] ds + 2\sigma^2 \int_0^t \mathbb{E}[(X_s^\beta - S_0^\beta)^2] ds. \end{aligned}$$

For any $x > 0$, $y \geq 0$ and $\frac{1}{2} \leq \beta < 1$, we have $|x^\beta - y^\beta| \leq |x - y|x^{\beta-1}$, see e.g. Lemma 2.2. in Cai and Wang [10]. Hence,

$$(150) \quad 2\sigma^2 \int_0^t \mathbb{E}[(X_s^\beta - S_0^\beta)^2] ds \leq 2\sigma^2 S_0^{2(\beta-1)} \int_0^t \mathbb{E}[(X_s - S_0)^2] ds = \sigma^2 S_0^{2(\beta-1)} \sigma^2 S_0^{2\beta} t^2.$$

Moreover, for $S_0 > \delta > 0$,

$$(151) \quad \begin{aligned} 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2] ds \\ = 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s \geq \delta}] ds + 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s < \delta}] ds. \end{aligned}$$

On the one hand,

$$(152) \quad 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s \geq \delta}] ds \leq 2\sigma^2 \delta^{2(\beta-1)} \int_0^t \mathbb{E}[(\hat{S}_s - X_s)^2] ds.$$

On the other hand,

$$(153) \quad \begin{aligned} 2\sigma^2 \int_0^t \mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^2 1_{X_s < \delta}] ds \\ \leq 2\sigma^2 \int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]} \sqrt{\mathbb{P}(X_s < \delta)} ds \\ \leq 2\sigma^2 \max_{0 \leq s \leq t} \sqrt{\mathbb{P}(X_s < \delta)} \int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]} ds. \end{aligned}$$

Note that

$$(154) \quad \int_0^t \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]} ds \leq \int_0^t \sqrt{4\mathbb{E}[\hat{S}_s^{4\beta} + X_s^{4\beta}]} ds,$$

and we can compute $\mathbb{E}[\hat{S}_s^{4\beta}]$ and $\mathbb{E}[X_s^{4\beta}]$ explicitly since \hat{S}_t is a CEV process and X_t is a Brownian motion. It is therefore easy to check that $\int_0^T \sqrt{\mathbb{E}[(\hat{S}_s^\beta - X_s^\beta)^4]} ds = O(T)$. Furthermore,

$$(155) \quad 2\sigma^2 \max_{0 \leq s \leq t} \sqrt{\mathbb{P}(X_s < \delta)} = 2\sigma^2 \Phi \left(\frac{\delta - S_0}{\sigma S_0^\beta \sqrt{t}} \right),$$

where $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$. Hence, by Gronwall's inequality, we conclude that

$$(156) \quad \mathbb{E}[(\hat{S}_T - X_T)^2] = O(T^2).$$

Note that $\hat{S}_t - X_t$ is a martingale. By Doob's martingale inequality,

$$(157) \quad \mathbb{E} \left[\max_{0 \leq t \leq T} |\hat{S}_t - X_t| \right] \leq C \sqrt{\mathbb{E}[(\hat{S}_T - X_T)^2]} = O(T).$$

Therefore, we conclude that

$$(158) \quad \begin{aligned} C(T) &= \mathbb{E} \left[\left(\frac{1}{T} \int_0^T X_t dt - S_0 \right)^+ \right] + O(T) \\ &= \mathbb{E} \left[\left(\sigma S_0^\beta \frac{1}{T} \int_0^T W_t dt \right)^+ \right] + O(T) \\ &= \sigma S_0^\beta \frac{\sqrt{T}}{\sqrt{3}} \mathbb{E}[Z 1_{Z>0}] + O(T), \end{aligned}$$

where $Z \sim N(0, 1)$. Finally, we can compute that

$$(159) \quad \mathbb{E}[Z 1_{Z>0}] = \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}}.$$

Hence, we proved the desired result. \square

Proof of Proposition 7. We will define $\mathcal{I}_K(K, S_0)$ as the solution of the variational problem (27), obtained by replacing the inequality (28) with the equality constraint $\int_0^1 g(t) dt = K$. This is solved by considering the variational problem for the auxiliary functional

$$(160) \quad \Lambda[g] := \frac{1}{2\sigma^2} \int_0^1 \frac{(g'(t))^2}{g(t)^{2\beta}} dt - \lambda \left(\int_0^1 g(t) dt - K \right),$$

where λ is a Lagrange multiplier.

The solution of this variational problem satisfies the Euler-Lagrange equation

$$(161) \quad g''(t) = \beta \frac{[g'(t)]^2}{g(t)} - \lambda \sigma^2 (g(t))^{2\beta},$$

with initial condition $g(0) = S_0$ and transversality condition $g'(1) = 0$.

This equation can be simplified by the change of variable

$$(162) \quad g(t) = S_0 (y(t))^{\frac{1}{1-\beta}}.$$

Expressed in terms of $y(t)$, the Euler-Lagrange equation (161) becomes

$$(163) \quad y''(t) = C (y(t))^{\frac{\beta}{1-\beta}},$$

with $C := -\lambda \sigma^2 (1-\beta) S_0^{2\beta-1}$. The solution $y(t)$ satisfies the initial condition $y(0) = 1$ and transversality condition $y'(1) = 0$. The rate function is expressed in terms of this solution as

$$(164) \quad \mathcal{I}_K(K, S_0) = \frac{S_0^{2-2\beta}}{2\sigma^2(1-\beta)^2} \int_0^1 [y'(t)]^2 dt.$$

The constraint $\int_0^1 g(t) dt = K$ reads

$$(165) \quad \int_0^1 (y(t))^{\frac{1}{1-\beta}} dt = \frac{K}{S_0}.$$

The differential equation (163) is known as the Emden-Fowler equation. The exponent $\gamma := \frac{\beta}{1-\beta}$ satisfies $\gamma \geq 1$ for the cases considered here $\beta \in [\frac{1}{2}, 1)$. This equation can be reduced to a first order ODE by noting the conservation of the quantity

$$(166) \quad E := \frac{1}{2}[y'(t)]^2 - C(1-\beta)(y(t))^{\gamma+1}.$$

Taking into account the boundary condition $y'(1) = 0$ we get the relation

$$(167) \quad [y'(t)]^2 = 2C(1-\beta) \left([y(t)]^{\gamma+1} - y_1^{\gamma+1} \right),$$

where we denoted $y_1 := y(1)$.

We distinguish the two cases:

1. $C > 0$. This corresponds to $y'(t) < 0$ and $y(1) < y(0) = 1$. From (165) we get that this corresponds to $K < S_0$.

2. $C < 0$. This corresponds to $y'(t) > 0$ and $y(1) > y(0) = 1$. From (165) we get that this corresponds to $K > S_0$.

We consider the two cases separately.

Case 1. $C > 0$. We can express y_1 in terms of C using the relation

$$(168) \quad 1 = \int_0^1 dt = \int_{y_1}^{y(0)} \frac{dy}{y'} = \frac{1}{\sqrt{2C(1-\beta)}} \int_{y_1}^1 \frac{dy}{\sqrt{y^{\gamma+1} - y_1^{\gamma+1}}}.$$

This relation can be used to eliminate C in terms of y_1 as

$$(169) \quad C = \frac{1}{2(1-\beta)} [A^{(+)}(y_1)]^2,$$

where we defined the function

$$(170) \quad \begin{aligned} A^{(+)}(x) &:= \int_x^1 \frac{dy}{\sqrt{y^{\gamma+1} - x^{\gamma+1}}} \\ &= \frac{2x}{\gamma+1} \frac{\sqrt{1-x^{\gamma+1}}}{x^{\gamma+1}} {}_2F_1 \left(\frac{\gamma}{\gamma+1}, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x^{\gamma+1}} \right), \quad 0 < x \leq 1. \end{aligned}$$

The constraint (165) can be written equivalently using (167) as

$$(171) \quad \frac{K}{S_0} = \int_0^1 [y(t)]^{\gamma+1} dt = [y_1]^{\gamma+1} + \frac{1}{2C(1-\beta)} \int_0^1 [y'(t)]^2 dy.$$

The integral can be expressed by a change of variable as

$$(172) \quad \begin{aligned} \int_0^1 dy [y'(t)]^2 &= \int_{y(0)}^{y(1)} y' dy = \sqrt{2C(1-\beta)} \int_{y(1)}^1 \sqrt{y^{\gamma+1} - y_1^{\gamma+1}} dy \\ &= A^{(+)}(y(1)) B^{(+)}(y(1)), \end{aligned}$$

where we defined

$$(173) \quad \begin{aligned} B^{(+)}(x) &:= \int_x^1 \sqrt{y^{\gamma+1} - x^{\gamma+1}} dy \\ &= \frac{2x}{3(\gamma+1)} \frac{(1-x^{\gamma+1})^{3/2}}{x^{\gamma+1}} {}_2F_1 \left(\frac{\gamma}{\gamma+1}, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x^{\gamma+1}} \right), \quad 0 < x \leq 1. \end{aligned}$$

The integral (172) is the same as the integral appearing in the expression for the rate function (164).

In conclusion, the rate function $\mathcal{I}_K(K, S_0)$ for $K < S_0$ is given by

$$(174) \quad \mathcal{I}_K(K, S_0) = \frac{S_0^{2(1-\beta)}}{2\sigma^2(1-\beta)^2} A^{(+)}(y_1) B^{(+)}(y_1),$$

where $y_1 < 1$ is the solution of the equation

$$(175) \quad \frac{K}{S_0} = y_1^{\gamma+1} + \frac{B^{(+)}(y_1)}{A^{(+)}(y_1)}.$$

Case 2. $C < 0$. We can express $y(1)$ in terms of C using the relation

$$(176) \quad 1 = \int_0^1 dt = \int_{y(0)}^{y(1)} \frac{dy}{y'} = \frac{1}{\sqrt{-2C(1-\beta)}} \int_1^{y_1} \frac{dy}{\sqrt{y_1^{\gamma+1} - y^{\gamma+1}}}.$$

We can use this relation to eliminate $-C > 0$ in terms of y_1 as

$$(177) \quad -C = \frac{1}{2(1-\beta)} [A^{(-)}(y_1)]^2,$$

where we defined the function

$$(178) \quad \begin{aligned} A^{(-)}(x) &:= \int_1^x \frac{dy}{\sqrt{x^{\gamma+1} - y^{\gamma+1}}} \\ &= \frac{2x}{\gamma+1} \frac{\sqrt{x^{\gamma+1} - 1}}{x^{\gamma+1}} {}_2F_1\left(\frac{\gamma}{\gamma+1}, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x^{\gamma+1}}\right), \quad x \geq 1. \end{aligned}$$

The constraint (165) can be written equivalently using (167) as

$$(179) \quad \frac{K}{S_0} = \int_0^1 [y(t)]^{\gamma+1} dt = y_1^{\gamma+1} + \frac{1}{2C(1-\beta)} \int_0^1 [y'(t)]^2 dy.$$

The integral can be written by a change of variable as

$$(180) \quad \begin{aligned} \int_0^1 [y'(t)]^2 dy &= \int_{y(0)}^{y(1)} y' dy = \sqrt{-2C(1-\beta)} \int_1^{y_1} \sqrt{y_1^{\gamma+1} - y^{\gamma+1}} dy \\ &= A^{(-)}(y_1) B^{(-)}(y_1), \end{aligned}$$

where we defined

$$(181) \quad \begin{aligned} B^{(-)}(x) &:= \int_1^x \sqrt{x^{\gamma+1} - y^{\gamma+1}} dy \\ &= \frac{2x}{3(\gamma+1)} \frac{(x^{\gamma+1} - 1)^{3/2}}{x^{\gamma+1}} {}_2F_1\left(\frac{\gamma}{\gamma+1}, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x^{\gamma+1}}\right), \quad x \geq 1. \end{aligned}$$

This gives also the integral appearing in the expression for the rate function in (164).

In conclusion, the rate function $\mathcal{I}_K(K, S_0)$ for $K > S_0$ is given by

$$(182) \quad \mathcal{I}_K(K, S_0) = \frac{S_0^{2(1-\beta)}}{2\sigma^2(1-\beta)^2} A^{(-)}(y_1) B^{(-)}(y_1),$$

where $y_1 > 1$ is the solution of the equation

$$(183) \quad \frac{K}{S_0} = y_1^{\gamma+1} - \frac{B^{(-)}(y_1)}{A^{(-)}(y_1)}.$$

The integrals $A^{(\pm)}(x), B^{(\pm)}(x)$ have been evaluated in closed form in terms of the hypergeometric function ${}_2F_1(a, b; c; z)$, defined as

$$(184) \quad {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} \frac{(1-t)^{c-b-1}}{(1-tz)^a} dt.$$

The results can be simplified by changing the variable $y_1^{\gamma+1} = z$ and introducing the functions $a^{(\pm)}(z) := (1-\beta)A^{(\pm)}(y_1)$, and $b^{(\pm)}(z) := (1-\beta)B^{(\pm)}(y_1)$. \square

Proof of Proposition 8. (i) The extremum condition for φ is

$$(185) \quad \varphi_* - \frac{K}{S_0} = \frac{\mathcal{G}^{(-)}(\varphi_*)}{\mathcal{F}^{(-)}(\varphi_*)},$$

where

$$(186) \quad \mathcal{F}^{(-)}(\varphi) := 2 \frac{d}{d\varphi} \mathcal{G}^{(-)}(\varphi) = \frac{S_0^{1-\beta}}{\sigma} \int_1^\varphi \frac{dz}{z^\beta \sqrt{\varphi - z}}.$$

The equation (185) is identical with the equation (183) for y_1 , identifying $\varphi_* = y_1^{\gamma+1}$. Substituting (185) into (36) we have

$$(187) \quad \mathcal{I}_K(K, S_0) = \frac{1}{2} \mathcal{F}^{(-)}(\varphi_*) \mathcal{G}^{(-)}(\varphi_*).$$

This result is identical to (182) with the identification $\varphi_* = y_1^{\gamma+1}$.

(ii) The extremum condition for χ is

$$(188) \quad \frac{K}{S_0} - \chi_* = \frac{\mathcal{G}^{(+)}(\chi_*)}{\mathcal{F}^{(+)}(\chi_*)},$$

where

$$(189) \quad \mathcal{F}^{(+)}(\chi) := -2 \frac{d}{d\chi} \mathcal{G}^{(+)}(\chi) = \frac{S_0^{1-\beta}}{\sigma} \int_\chi^1 \frac{dz}{z^\beta \sqrt{z - \chi}}.$$

The equation (188) is identical with the equation (175) for y_1 , identifying $\chi_* = y_1^{\gamma+1}$. Substituting (188) into (38) we have

$$(190) \quad \mathcal{I}_K(K, S_0) = \frac{1}{2} \mathcal{F}^{(+)}(\chi_*) \mathcal{G}^{(+)}(\chi_*).$$

This result is identical to (174) with the identification $\chi_* = y_1^{\gamma+1}$. \square

Proof of Corollary 9. (i) follows from Lemma 29 in [47]. The technical conditions of this Lemma require that $\mathcal{G}^{(-)}(\varphi)$ is an increasing function and that $[\mathcal{G}^{(-)}(\varphi)]^2$ has superlinear growth as $\varphi \rightarrow \infty$. The first condition is satisfied as the derivative of $\mathcal{G}^{(-)}(\varphi)$ is given by (186), which is a positive function.

The second technical condition is also satisfied, as follows. Using the asymptotics of the hypergeometric function

$$(191) \quad {}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\varphi}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(1-\beta)}{\Gamma(\frac{5}{2}-\beta)} + O(\varphi^{-1}), \quad \text{as } \varphi \rightarrow \infty.$$

we get that

$$(192) \quad [\mathcal{G}^{(-)}(\varphi)]^2 \sim \frac{(\varphi-1)^3}{\varphi^{2\beta}}, \quad \text{as } \varphi \rightarrow \infty.$$

This has indeed superlinear growth provided that $\beta < 1$.

(ii) follows from Lemma 30 in [47]. This requires the following two technical conditions: $\mathcal{G}^{(+)}(\chi)$ is a decreasing function, and the infimum in (38) is not reached at the lower boundary $\chi = 0$. The first condition follows indeed from (189), as the integral in this expression is positive.

The second condition follows by noting that we have, for $\beta \geq \frac{1}{2}$

$$(193) \quad \lim_{\chi \rightarrow 0} \frac{d}{d\chi} \left(\frac{\frac{1}{2}[\mathcal{G}^{(+)}(\chi)]^2}{\frac{K}{S_0} - \chi} \right) = -\infty.$$

This is obtained by writing the derivative explicitly

$$(194) \quad \frac{d}{d\chi} \left(\frac{\frac{1}{2}[\mathcal{G}^{(+)}(\chi)]^2}{\frac{K}{S_0} - \chi} \right) = \frac{\mathcal{G}^{(+)}(\chi) \frac{d}{d\chi} \mathcal{G}^{(+)}(\chi)}{\frac{K}{S_0} - \chi} + \frac{1}{2} \frac{[\mathcal{G}^{(+)}(\chi)]^2}{(\frac{K}{S_0} - \chi)^2}.$$

Furthermore, the functions appearing here have the $\chi \rightarrow 0$ limits, for $\beta \geq \frac{1}{2}$,

$$(195) \quad \mathcal{G}^{(+)}(\chi) = 1 + O\left(\chi^{\frac{3}{2}-\beta}\right), \quad \text{as } \chi \rightarrow 0$$

and

$$(196) \quad \frac{d}{d\chi} \mathcal{G}^{(+)}(\chi) = -\infty \quad \text{as } \chi \rightarrow 0.$$

The relation (195) follows from the $\chi \rightarrow 0$ asymptotics of the hypergeometric function, which can be extracted from Equation (208)

$$(197) \quad {}_2F_1\left(\frac{3}{2}, \beta; \frac{5}{2}; 1 - \frac{1}{\chi}\right) = \frac{3}{3-2\beta} \chi^\beta + \frac{\Gamma(\frac{5}{2})\Gamma(\beta-\frac{3}{2})}{\Gamma(\beta)} \chi^{3/2}.$$

The relation (196) is obtained from (189) by noting that the integral on the RHS is bounded from below as

$$(198) \quad \int_\chi^1 \frac{dz}{z^\beta \sqrt{z-\chi}} \geq \int_\chi^1 dz z^{-\frac{1}{2}-\beta} = \frac{1}{\frac{1}{2}-\beta} (1 - \chi^{\frac{1}{2}-\beta}) \rightarrow +\infty, \quad \chi \rightarrow 0_+.$$

In the last step we used $\beta > \frac{1}{2}$. The conclusion holds also for $\beta = \frac{1}{2}$, using the relation

$$(199) \quad \int_\chi^1 \frac{dz}{\sqrt{z(z-\chi)}} = 2 \log(\sqrt{1-\chi} + 1) - \log \chi \rightarrow \infty, \quad \chi \rightarrow 0_+.$$

This shows that the infimum in (38) is not reached at the lower boundary $\chi = 0$. This justifies the application of Lemma 30 in [47].

iii) The conclusion follows immediately from the result for the rate function $\mathcal{I}(K, S_0)$ given by Theorem 5 and the monotonicity properties of $\mathcal{I}_K(K, S_0)$ proven above in (i) and (ii). \square

Proof of Proposition 12. We give here the proof for the large-strike asymptotics of the rate function $\mathcal{I}(K, S_0)$.

For this case we are interested in the $x \rightarrow \infty$ asymptotics of the functions $a^{(-)}(x), b^{(-)}(x)$. For this purpose it is useful to transform the argument $z = 1 - \frac{1}{x}$ of the hypergeometric functions appearing in the expressions of these functions as

$$(200) \quad z \rightarrow 1 - z = \frac{1}{x}$$

using the identity 15.3.6 in Abramowitz and Stegun [1].

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; c-a-b+1; 1-z). \end{aligned}$$

We get, for $\beta \in [\frac{1}{2}, 1)$,

$$(201) \quad {}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right) = \frac{\Gamma(\frac{3}{2})\Gamma(1-\beta)}{\Gamma(\frac{3}{2}-\beta)} + O(x^{\beta-1}),$$

$$(202) \quad {}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right) = \frac{\Gamma(\frac{5}{2})\Gamma(1-\beta)}{\Gamma(\frac{5}{2}-\beta)} + O(x^{\beta-1}),$$

as $x \rightarrow \infty$.

The solution of the equation (33) for x for $K/S_0 \gg 1$ is

$$(203) \quad x = \frac{3-2\beta}{2(1-\beta)} \left(\frac{K}{S_0}\right) + O(K/S_0).$$

Substituting x into the expression for the rate function of Proposition 7 we obtain the large-strike asymptotics of $\mathcal{I}(K, S_0)$ given in Proposition 12. \square

Proof of Proposition 13. We give here the proof for the small-strike asymptotics of the rate function $\mathcal{I}(K, S_0)$.

We require the $x \rightarrow 0_+$ asymptotics for $a^{(+)}(x), b^{(+)}(x)$. This is obtained by changing the $z = 1 - \frac{1}{x}$ argument of the hypergeometric functions appearing in the expressions for these functions as

$$(204) \quad z \rightarrow \frac{1}{z-1} = -x,$$

using the identity 15.3.8 in Abramowitz and Stegun [1]

$$\begin{aligned} (205) \quad {}_2F_1(a, b; c; z) &= (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} {}_2F_1\left(a, c-b; a-b+1; \frac{1}{z-1}\right) \\ &\quad + (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, c-a; b-a+1; \frac{1}{z-1}\right). \end{aligned}$$

This can be used to find the asymptotics for $x \rightarrow 0_+$, together with the small- x asymptotics

$$(206) \quad {}_2F_1(a, c-b; a-b+1; x) = 1 + O(x).$$

We get

$$\begin{aligned}
 (207) \quad {}_2F_1\left(\beta, \frac{1}{2}; \frac{3}{2}; 1 - \frac{1}{x}\right) &= x^\beta \frac{\Gamma(3/2)\Gamma(1/2 - \beta)}{\Gamma(1/2)\Gamma(3/2 - \beta)}(1 + O(x)) \\
 &\quad + x^{\frac{1}{2}} \frac{\Gamma(3/2)\Gamma(\beta - 1/2)}{\Gamma(\beta)}(1 + O(x)) \\
 &= x^\beta \frac{1}{1 - 2\beta}(1 + O(x)) + x^{\frac{1}{2}} \frac{\Gamma(3/2)\Gamma(\beta - 1/2)}{\Gamma(\beta)}(1 + O(x)),
 \end{aligned}$$

and

$$\begin{aligned}
 (208) \quad {}_2F_1\left(\beta, \frac{3}{2}; \frac{5}{2}; 1 - \frac{1}{x}\right) &= x^\beta \frac{\Gamma(5/2)\Gamma(3/2 - \beta)}{\Gamma(3/2)\Gamma(5/2 - \beta)}(1 + O(x)) \\
 &\quad + x^{\frac{3}{2}} \frac{\Gamma(5/2)\Gamma(\beta - 3/2)}{\Gamma(\beta)}(1 + O(x)) \\
 &= x^\beta \frac{3}{3 - 2\beta}(1 + O(x)) + x^{\frac{3}{2}} \frac{\Gamma(5/2)\Gamma(\beta - 3/2)}{\Gamma(\beta)}(1 + O(x)).
 \end{aligned}$$

For $\frac{1}{2} < \beta < 1$, the dominant term in these expansions as $x \rightarrow 0_+$ is the second term in (207), and the first term in (208).

The equation for x as $K \rightarrow 0$ becomes approximatively

$$(209) \quad \frac{K}{S_0} = x^{\beta - \frac{1}{2}} \frac{\Gamma(\beta)}{\sqrt{\pi}(\frac{3}{2} - \beta)\Gamma(\beta - \frac{1}{2})} + O(x).$$

Substituting x into the expression for the rate function of Proposition 7 we obtain the small-strike asymptotics of $\mathcal{I}(K, S_0)$ given in Proposition 13. \square

6.4. Proof of the results in Section 4.

Proof of Theorem 14. For any $\theta \in \mathbb{R}$, $\mathbb{E}[e^{\frac{\theta}{T^2} \int_0^t S_s ds - \frac{\theta\kappa}{T} S_T} | S_0] = e^{A(T; \frac{\theta}{T^2}, -\frac{\theta\kappa}{T}) S_0}$, where $A(t; \theta; \phi)$ satisfies the ODE:

$$(210) \quad A'(t; \theta, \phi) = (r - q)A(t; \theta, \phi) + \frac{1}{2}\sigma^2 A(t; \theta, \phi)^2 + \theta,$$

with $A(0; \theta, \phi) = \phi$.

For $\theta > 0$,

$$\begin{aligned}
 (211) \quad A(t; \theta, \phi) &= \frac{\sqrt{2\sigma^2\theta - (r - q)^2}}{\sigma^2} \tan \left[\frac{\sqrt{2\sigma^2\theta - (r - q)^2}}{2} t + \tan^{-1} \left(\frac{r - q + \sigma^2\phi}{\sqrt{2\sigma^2\theta - (r - q)^2}} \right) \right] \\
 &\quad - \frac{r - q}{\sigma^2},
 \end{aligned}$$

and for $\theta < 0$,

$$(212) \quad A(t; \theta, \phi) = \frac{(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}})e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}}{(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi) - e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)} \\ - \frac{(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}})(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)}{(\frac{r-q}{\sigma^2} + \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi) - e^{t\sqrt{(r-q)^2 - 2\theta\sigma^2}}(\frac{r-q}{\sigma^2} - \sqrt{\frac{(r-q)^2}{\sigma^4} - \frac{2\theta}{\sigma^2}} + \phi)}.$$

For $0 \leq \theta < \theta_c$,

$$(213) \quad \lim_{T \rightarrow 0} TA \left(T; \frac{\theta}{T^2}, \frac{-\kappa\theta}{T} \right) = \sqrt{\frac{2\theta}{\sigma^2}} \tan \left(\sqrt{\frac{\sigma^2\theta}{2}} + \tan^{-1} \left(-\sigma\kappa\sqrt{\frac{\theta}{2}} \right) \right),$$

and this limit is ∞ if $\theta \geq \theta_c$, where θ_c is the unique positive solution to the equation:

$$(214) \quad \sqrt{\frac{\sigma^2\theta_c}{2}} + \tan^{-1} \left(-\sigma\kappa\sqrt{\frac{\theta_c}{2}} \right) = \frac{\pi}{2}.$$

To see that (214) has a unique positive solution, let us define:

$$(215) \quad F(x) := \sqrt{\frac{\sigma^2}{2}}x + \tan^{-1} \left(-\sigma\kappa\frac{1}{\sqrt{2}}x \right) - \frac{\pi}{2}.$$

Then, $F(0) = -\frac{\pi}{2}$ and $F(\infty) = \infty$. On the other hand, we can compute that

$$(216) \quad F'(x) = \sqrt{\frac{\sigma^2}{2}} - \frac{\sigma\kappa}{\sqrt{2}} \frac{1}{\frac{1}{2}\sigma^2\kappa^2x^2 + 1}, \quad F''(x) = \frac{\sigma\kappa}{\sqrt{2}} \frac{\sigma^2\kappa^2x}{(\frac{1}{2}\sigma^2\kappa^2x^2 + 1)^2}.$$

Since $F''(x) > 0$ for any $x > 0$, and $F(0) = -\frac{\pi}{2} < 0$ and $F(\infty) = \infty$, it follows that $F(x) = 0$ has a unique positive solution.

For $\theta < 0$,

$$(217) \quad \lim_{T \rightarrow 0} TA \left(T; \frac{\theta}{T^2}, \frac{-\kappa\theta}{T} \right) = -\frac{\sqrt{-2\theta}}{\sigma} \frac{\frac{\sqrt{-2\theta}}{\sigma}(e^{\sigma\sqrt{-2\theta}} - 1) + \theta\kappa(1 + e^{\sigma\sqrt{-2\theta}})}{\frac{\sqrt{-2\theta}}{\sigma}(1 + e^{\sigma\sqrt{-2\theta}}) - \theta\kappa(1 - e^{\sigma\sqrt{-2\theta}})} \\ = -\frac{\sqrt{-2\theta}}{\sigma} \frac{\frac{\sqrt{-2\theta}}{\sigma} + \theta\kappa}{\frac{\sqrt{-2\theta}}{\sigma} - \theta\kappa} \frac{e^{\sigma\sqrt{-2\theta}} - 1}{e^{\sigma\sqrt{-2\theta}} + 1} \\ = -\frac{\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2}\sqrt{-2\theta} + \tanh^{-1} \left(-\sigma\kappa\sqrt{\frac{-\theta}{2}} \right) \right).$$

Therefore,

$$(218) \quad \Lambda(\theta) := \lim_{T \rightarrow 0} T \log \mathbb{E} \left[e^{\frac{\theta}{T^2} \int_0^T S_t dt - \frac{\theta}{T} \kappa S_T} \right]$$

$$= \begin{cases} \frac{\sqrt{2\theta}}{\sigma} \tan \left(\frac{\sigma}{2} \sqrt{2\theta} + \tan^{-1} \left(-\sigma \kappa \sqrt{\frac{\theta}{2}} \right) \right) S_0 & \text{if } 0 \leq \theta < \theta_c \\ -\frac{\sqrt{-2\theta}}{\sigma} \tanh \left(\frac{\sigma}{2} \sqrt{-2\theta} + \tanh^{-1} \left(-\sigma \kappa \sqrt{\frac{-\theta}{2}} \right) \right) S_0 & \text{if } \theta \leq 0 \\ +\infty & \text{otherwise} \end{cases}.$$

It is easy to show that $\Lambda_f(\theta)$ is differentiable for any $\theta < \theta_c$ and $\Lambda'_f(\theta) \rightarrow \infty$ as $\theta \uparrow \theta_c$. Hence, $\mathbb{P} \left(\frac{1}{T} \int_0^T S_t dt - \kappa S_T \in \cdot \right)$ satisfies a large deviation principle with the rate function $\mathcal{I}_f(\kappa, S_0)$ given in (65) by applying the Gärtner-Ellis theorem, see Theorem 20 in the Appendix. \square

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REFERENCES

- [1] Abramowitz, M. and I. A. Stegun (1972). Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications, New York.
- [2] Alòs, E., Léon, J. and J. Vives. (2007). On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility. *Finance and Stochastics*. **11**, 571-589.
- [3] Alziary, B., Decamps, J. P. and P. F. Koehl, A PDE approach to Asian options: Analytical and Numerical evidence. *Journal of Banking and Finance*. **21**, 613-640 (1997).
- [4] Andersen, L. and J. Andreasen (2000). Volatility skews and extensions of the Libor market model. *Appl. Math. Finance* **7**, 1-32.
- [5] Andersen, L. and A. Lipton. (2013). Asymptotics for exponential Lévy processes and their volatility smile: Survey and new results. *International Journal of Theoretical and Applied Finance*. **16**, 1350001.
- [6] Armstrong, J., Forde, M., Lorig, M. and H. Zhang. (2016) Small-Time Asymptotics under Local-Stochastic Volatility with a Jump-to-Default Curvature and the Heat Kernel Expansion. *SIAM J. Fin. Math.* **8** 82-113.
- [7] Baldi, P. and L. Caramellino. (2011). General Freidlin-Wentzell large deviations and positive diffusions. *Statistics and Probability Letters*. **81**, 1218-1229.
- [8] Berestycki, H., Busca, J. and I. Florent. (2002). Asymptotics and calibration of local volatility models. *Quantitative Finance*. **2**, 61-69.
- [9] Berestycki, H., Busca, J. and I. Florent. (2004). Computing the implied volatility in stochastic volatility models. *Commun. Pure Appl. Math.* **57**, 1352-1373.
- [10] Cai, Y. and S. Wang. (2015). Central limit theorem and moderate deviation principle for CKLS model with small random perturbation. *Stat. Prob. Letters* **98**, 6-11.
- [11] Carr, P. and M. Schröder. (2003). Bessel processes, the integral of geometric Brownian motion, and Asian options. *Theory of Probability and its Applications*. **48**, 400-425.
- [12] Carr, P. and J. Sun. (2007). A new approach for option pricing under stochastic volatility. *Review of Derivatives Research*. **10**(2), 87-150.
- [13] Carr, P. and L. Wu (2003). What type of process underlies options? A simple robust test. *The Journal of Finance*. **58**(6), 2581-2610.
- [14] Cheng, W., Costanzino, N., Liechty, J., Mazzucato, A. and V. Nistor. (2011). Closed-form asymptotics and numerical approximations of 1D parabolic equations with applications to option pricing. *SIAM J. Fin. Math.* **2**, 901-934.
- [15] Cox, J. C. (1996). Notes on Option Pricing I: Constant elasticity of variance diffusions. Reprinted in the *Journal of Portfolio Management*. **23**, 15-17.

- [16] Cox, J. C., J. E. Ingersoll and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica*. **53**(2), 385-407.
- [17] Cox, J. C. and S. A. Ross. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*. **3**, 145-166.
- [18] Dassios, A. and J. Nagradjasarma. (2006) The square-root process and Asian options. *Quant. Finance* **6**, 337-347.
- [19] Dembo, A. and O. Zeitouni. *Large Deviations Techniques and Applications*. 2nd Edition, Springer, New York, 1998.
- [20] Donati-Martin, C., Rouault, A. M. Yor, and M. Zani. (2004). Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields* **129**, 261-289.
- [21] Drimus, G.G. (2012). Options on realized variance by transform methods: A non-affine stochastic volatility model, *Quantitative Finance* **12**, 1679-1694.
- [22] Dufresne, D. (2000). Laguerre series for Asian and other options. *Math. Finance* **10**, 407-428.
- [23] Dufresne, D. (2001). The integrated square-root process. Technical report, University of Montreal.
- [24] Dufresne, D. Bessel processes and a functional of Brownian motion, in M. Michele and H. Ben-Ameur (Ed.), *Numerical Methods in Finance*, 35-57, Springer, 2005.
- [25] Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*. **54**, 173.
- [26] Feng, J., Forde, M. and J.-P. Fouque. (2010). Short maturity asymptotics for a fast mean-reverting Heston stochastic volatility model. *SIAM Journal on Financial Mathematics*. **1**, 126-141.
- [27] Feng, J., Fouque, J.-P. and R. Kumar. (2012). Small-time asymptotics for fast mean-reverting stochastic volatility models. *Annals of Applied Probability*. **22**, 1541-1575.
- [28] Figueroa-López, J. E. and M. Forde. (2012). The small-maturity smile for exponential Lévy models. *SIAM J. Finan. Math.* **3**, 33-65.
- [29] Forde, M. and A. Jacquier. (2009). Small time asymptotics for implied volatility under the Heston model. *International Journal of Theoretical and Applied Finance*. **12**, 861.
- [30] Forde, M. and A. Jacquier. (2011). Small time asymptotics for an uncorrelated Local-Stochastic volatility model. *Applied Mathematical Finance*. **18**, 517-535.
- [31] Forde, M., Jacquier, A. and R. Lee. (2012). The small-time smile and term structure of implied volatility under the Heston model. *SIAM J. Finan. Math.* **3**, 690-708.
- [32] Foschi, P., Pagliarani, S. and A. Pascucci. (2013). Approximations for Asian options in local volatility models. *Journal of Computational and Applied Mathematics*. **237**, 442-459.
- [33] Fu, M., Madan, D. and T. Wang. (1998). Pricing continuous time Asian options: a comparison of Monte Carlo and Laplace transform inversion methods. *J. Comput. Finance* **2**, 49-74.
- [34] Fusai, M., Marena, M. and A. Roncoroni. (2008). Analytical pricing of discretely monitored Asian-style options: Theory and applications to commodity markets. *J. of Banking and Finance* **32**, 2033-2045.
- [35] Gao, K. and R. Lee. (2014). Asymptotics of implied volatility to arbitrary order. *Finance and Stochastics*. **18**, 349-392.
- [36] Gatheral, J., Hsu, E. P., Laurent, P., Ouyang, C. and T.-H. Wang. (2012). Asymptotics of implied volatility in local volatility models. *Mathematical Finance*. **22**, 591-620.
- [37] Gatheral, J. and T.-H. Wang. (2012). The heat-kernel most-likely-path approximation. *IJTAF*. **15**, 1250001.
- [38] Geman, H. and M. Yor. (1993). Bessel processes, Asian options and perpetuities. *Math. Finance* **3**, 349-375.
- [39] Gobet, E. and M. Miri. (2014). Weak approximation of averaged diffusion processes. *Stoch. Proc. Appl.* **124**, 475-504.
- [40] Hagan, P. and D. Woodward. (1999). Equivalent Black volatilities. *Applied Mathematical Finance* **6**, 147-157.
- [41] Henderson, V. and R. Wojakowski (2002). On the equivalence of floating and fixed-strike Asian options. *J. Appl. Prob.* **39**, 391-394.
- [42] Henry-Labordère, P. (2005). Analysis, Geometry and Modeling in Finance: Advanced Methods in Option Pricing. Chapman and Hall/CRC Financial Mathematical Series, 2008.
- [43] Linetsky, V. (2004). Spectral expansions for Asian (Average price) options. *Operations Research* **52**, 856-867.
- [44] Linetsky, V. and R. Mendoza (2009). Constant Elasticity of Variance (CEV) Diffusion Model. in *Encyclopedia of Quantitative Finance*, Ed. Rama Cont.

- [45] Mazzon, A. (2011). Processo square root. PhD Thesis, University of Bologna.
 - [46] Muhle-Karbe, J. and M. Nutz. (2011). Small-time asymptotics of option prices and first absolute moments. *J. Appl. Prob.* **48**, 1003-1020.
 - [47] Pirjol, D. and L. Zhu (2016). Short maturity Asian options in local volatility models. *SIAM J. Fin. Math.* **7**, 947-992.
 - [48] Rogers, L. and Z. Shi. (1995). The value of an Asian option. *J. Appl. Prob.* **32**, 1077-1088.
 - [49] Shiraya, K., A. Takahashi and M. Toda. (2011). Pricing barrier and average options under stochastic volatility environment. *J. Comput. Finance* **15**, 111-148.
 - [50] Tankov, P. Pricing and hedging in exponential Lévy models: Review of recent results. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.* pages 319-359, Springer, Berlin, 2010.
 - [51] Varadhan, S. R. S. (1967). Diffusion processes in a small time interval. *Communications on Pure and Applied Mathematics*. **20**, 659-685.
 - [52] Varadhan, S. R. S. *Large Deviations and Applications*, SIAM, Philadelphia, 1984.
 - [53] Vecer, J. (2001). A new PDE approach for pricing arithmetic average Asian options. *J. Comput. Finance*. **4**, 105-113.
 - [54] Vecer, J. and M. Xu. (2002). Unified Asian pricing. *Risk*. **15**, 113-116.
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